Partition Relation Perspectives

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Menachem's doctoral work

In 1973, Menachem Magidor received his PhD under the direction of Azriel Levy at the Hebrew University of Jerusalem.

First memory of Menachem $\sim$ 1973

Hebrew University of Jerusalem

Menachem spent most of his career at the Hebrew University of Jerusalem.

- He was promoted to Associate Professor in 1978
- He was promoted to Professor 1982
- He served as president of the university 1997-2009

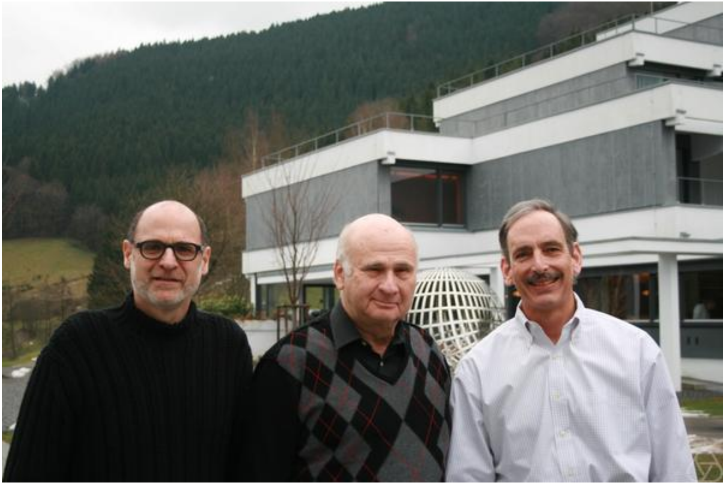
Menachem's students (Math Genealogy Project)

- Charly Bitton, Hebrew Univerity, 1998
- Moti Gitik, Hebrew University, 1980 (A. Levy + MM)
- Hiam Judah, Hebrew University, 1987
- Yechiel Kimchi, Hebrew University, 1987
- Gregory Lafitte, Ecole Normal Superieure de Lyon, 2002
- Amir Leshem, Hebrew University, 1998
- Ami Litman, Hebrew University, 1981
- Miri Segal, Hebrew University, 1997
- Anna Sfard, Hebrew University, 1989
- Huiling Zhu, National University of Singapore, 2012

Some co-authors (MathSciNet)

Uri Abraham	Ron Aharoni	Arthur Apter
Joan Bagaria	Shai Ben-David	Rachel Eliyahu-Zohary
Maxim Burke	Brent Cody	James Cummings
Paul Erdős	Ilijas Farah	Qi Feng
Matthew Foreman	Sy-David Friedman	László Fuchs
Fred Galvin	Moti Gitik	Yuri Gurevic
Thomash Jech	Istvan Juhász	Akihiro Kanamori
Richard Ketchersid	Sarit Kraus	Paul Larson
Daniel Lehmann	Amir Leshem	Jean-Pierre Levinski
Jerome Malitz	Willam Mitchell	Gadi Moran
Karel Prikry	John Rosenthal	Matayahu Rubin
Hiroshi Sakai	Ralf Schindler	Karl Schlechta
Saharon Shelah	Richard Shore	Gabriel Srour
Jonathan Stav	Jouko Vaananen	W. Hugh Woodin

Thanks for all the conferences you worked on



The conversation shifts to mathematics

Balanced BHT Theorem

Balanced Baumgartner-Hajnal-Todorcevic (BHT) Theorem, 1993

For every regular uncountable κ, $\xi < \kappa$, and $k < \omega$,

$$(2^\kappa)^+ \to (\kappa + \xi)^2_k.$$

This theorem has a very nice proof using ideals that come from chains of elementary submodels.

Balanced BHT Theorem

Ari Brodsky's Balanced BHT Theorem for Trees, 2014

For every infinite regular cardinal κ, ξ *with* $2^{|\xi|} < \kappa$, *and* $k < \omega$,

$$\text{non-}\left(2^{<\kappa}\right)\text{-special tree } \to (\kappa + \xi)^2_k.$$

Brodsky developed a theory of stationary subsets of trees in order to prove this theorem and also used ideals that come from chains of elementary submodels.

Polarized Partitions and Cardinal Invariants

Theorem (Paul Erdős and Richard Rado, 1956)

The Continuum Hypothesis implies $\begin{pmatrix} \aleph_1 \\ \omega \end{pmatrix} \nrightarrow \begin{pmatrix} \aleph_1 \\ \omega \end{pmatrix}^{1,1}_2$

Here $\begin{pmatrix} \theta \\ \omega \end{pmatrix} \rightarrow \begin{pmatrix} \theta \\ \omega \end{pmatrix}^{1,1}_2$ means for every function $c : \theta \times \omega \rightarrow 2$ there are $A \subseteq \theta$ and $B \subseteq \omega$ such that $|A| = \theta$, B is infinite and $c \restriction (A \times B)$ is constant.

Polarized Partitions and Cardinal Invariants

Theorem (Shimon Garti and Saharon Shelah, 2014)

It is consistent with ZFC, that all cardinals θ with $\aleph_1 \leq \theta \leq 2^\omega$,

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix} \to \begin{pmatrix} \theta \\ \omega \end{pmatrix}^{1,1}_2$$

Earlier they had shown the partition relation was true for $\theta < \mathfrak{s}$ (*splitting number*) and here show it is true for $\mathfrak{r} < \theta$ (*reaping number*).
Their model has $\aleph_1 = \mathfrak{r} =< \mathfrak{s} = \aleph_2 = \mathfrak{c}$.
Question: is it consistent that $\mathfrak{r} < \mathfrak{s}$ and $\mathfrak{c} > \aleph_2$?

Topological reduction of colors

Given countable ordinals α and β, Claribet Piña has investigated the minimum number of colors m needed, so that if pairs from α are colored with finitely many colors (at least 2), then there is a topological copy of β whose pairs realize at most m colors.

This may be expressed compactly as $(\forall \ell > 1)\left[\alpha \to_{top} (\beta)^2_{\ell,m}\right]$.

Theorem (Piña 2015)

For all $\ell > 1$,

$$\omega^\omega \to_{top} (\omega^2 + 1)^2_{\ell,6},$$

but

$$\omega^\omega \nrightarrow_{top} (\omega^2 + 1)^2_{9,5}.$$

Topological Pigeonhole Principle

Building on the ordinal pigeonhold principle of Eric Milner and Richard Rado which calculates $P^{\mathrm{ord}}(\alpha_i)$, and work of Bill Weiss, of Prikry and Solovay, and of Shelah, Jacob Hilton constructed a pigeonhole principle for ordinal topological spaces. Here is the theorem he describes as his main breakthrough.

Theorem (Jacob Hilton, JSL, to appear)

Suppose $0 < \alpha_1, \alpha_2, \ldots, \alpha_k < \omega_1$.

1. $P^{top}(\omega^{\alpha_1} + 1, \omega^{\alpha_2} + 1, \ldots, \omega^{\alpha_k} + 1) = \omega^{\alpha_1 \# \alpha_2 \# \ldots \# \alpha_k} + 1$.
2. $P^{top}(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_k}) = \omega^{P^{ord}(\alpha, \alpha_2, \ldots, \alpha_k)}$.

If $\alpha = \omega^{a_0} \cdot m_0 + \cdots + \omega^{a_p} \cdot m_p$ and $\beta = \omega^{a_0} \cdot n_0 + \cdots + \omega^{a_p} \cdot n_p$ where $a_0 > a_1 > \cdots > a_p$ and $m_0, m_1, \ldots m_p, n_0, n_1, \ldots n_p < \omega$, then the Hessenberg sum (natural sum) is defined by
$\alpha \# \beta = \omega^{a_0} \cdot (m_0 + n_0) + \cdots + \omega^{a_p} \cdot (m_p + n_p)$

Topological Ramsey numbers

In their search for Ramsey numbers of ordinal topological spaces, Andres Caicedo and Jacob Hilton looked at two kinds of goals: R^{top} when the goals were subspaces homeomorphic to given ordinal spaces and R^{cl} when the goals that were subspaces both order-isomorphic to given ordinal spaces and closed in their own supremum.

Caicedo and Hilton, building on Hilton's work, proved an analogous closed pigeonhole principle, $P^{cl}(\alpha_i)_{i<\kappa}$.

Topological Ramsey numbers

They write $R^{top}(\alpha_i)_{i<\kappa}$ for the topological Ramsey number, namely the least ordinal β such that

$$\beta \rightarrow_{top} (\alpha_i)^2_{i<\kappa}$$

Similarly, they write $R^{cl}(\alpha_i)_{i<\kappa}$ for the closed Ramsey number, namely the least ordinal β such that

$$\beta \rightarrow_{cl} (\alpha_i)^2_{i<\kappa}$$

Topological Ramsey numbers

Paul Erdős and Eric Milner proved the following theorem for ordinal partition relations:

Theorem (Erdős, Milner)

If $\omega^{\alpha} \to (\omega^{1+\beta}, k)^2$*, then* $\omega^{\alpha+\beta} \to (\omega^{1+\beta}, 2k)^2$*.*

Andres Caicedo and Jacob Hilton found a counterpart which they call
the Weak Topological Erdős-Milner Theorem.

Theorem (Caicedo, Hilton)

Let α *and* β *be countable non-zero ordinals, and let* $k > 1$ *be a positive integer. If*

$$\omega^{\omega^{\alpha}} \to_{top} (\omega^{\beta}, k)^2,$$

then

$$\omega^{\omega^{\alpha}\cdot\beta} \to_{top} (\omega^{\beta}, k+1)^2,$$

Topological Ramsey numbers

In a corollary to their main theorem, Caicedo and Hilton prove these Ramsey numbers coincide in certain cases and give upper bounds for them.

Theorem (Caicedo, Hilton)

Suppose $0 < \alpha < \omega_1$ and k, m and n are positive integers.

1. $R^{top}(\alpha, k)$ *is countable.*
2. $R^{top}(\omega^{\omega^\alpha}, k+1) = R^{cl}(\omega^{\omega^\alpha}, k+1) \leq \omega^{\omega^{\alpha} \cdot k}$.
3. $R^{top}(\omega^{\omega^\alpha} + 1, k+1) = R^{cl}(\omega^{\omega^\alpha} + 1, k+1)$ *and the common value is bounded by* $\omega^{\omega^{\alpha} \cdot k} + 1$ *if α is infinite, and bounded by* $\omega^{\omega^{(n+1) \cdot k - 1}} + 1$ *if $\alpha = n$ is finite.*

Topological Ramsey Theory

Question: is there a topological partition ordinal $\alpha > \omega$?

That is, is there $\alpha > \omega$ for which $\alpha \rightarrow_{top} (\alpha, 3)^2$?

Question: is it possible to reduce the computation of $R^{top}(\alpha, k)$ to finite combinatorial problems, even for $\alpha < \omega^2$?

Halpern-Läuchli Theorem

Copies of illustrations of a k-dense subset and a k-x-dense subset from *Ramsey Theory* by Todorcevic:

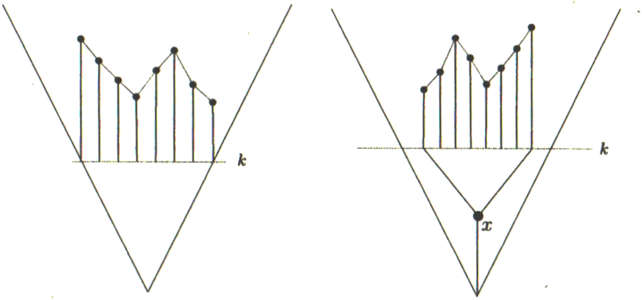


Figure 3.2 A k-dense set and a k-x-dense set.

A *k-dense matrix* (*k-$\vec{x}$-dense matrix*) for $\prod_{i<d} T_i$ is $\prod_{i<d} X_i$ where each X_i is a k-dense matrix (k-x_i-dense matrix).

Halpern-Läuchli Theorem

(Asymmetric Version of the Halpern-Läuchli Theorem)

For every coloring of the finite product of finitely branching trees, $\prod_{i<d} T_i = K_0 \cup K_1$, either

- K_0 contains a k-dense matrix for every integer k, or
- there is some $\vec{x} \in \prod_{i<d} T_i$ so that K_1 contains a k-$\vec{x}$-dense matrix for every interger k.

Halpern-Läuchli Theorem

Laver's Conjecture:

For every (level set) coloring of the infinite product of downwards closed perfect $\prod_{i<\omega} T_i = K_0 \cup K_1$, there is some $j < 2$ and some $\vec{x} \in \prod_{i<\omega} T_i$ such that for each $k < \omega$, K_j contains a k-$\vec{x}$-dense (level set) matrix.

Joint work with Džamonja and Mitchell

Theorem (Denis Devlin, 1979)

$$\mathbb{Q} \to (\mathbb{Q})^n_{<\omega,t_n} \text{ and } \mathbb{Q} \not\to (\mathbb{Q})^n_{<\omega,t_n-1},$$

where t_n is the nth tangent number.

For regular cardinals κ, there is a natural order $\leq_Q$ extending lexicographic order on ${}^{\kappa>}2$ such that $\mathbb{Q}_\kappa = \langle {}^{\kappa>}2, \leq_Q \rangle$ is a κ-dense linear order.

Joint work with Džamonja and Mitchell

Theorem (Džamonja, Larson, Mitchell, 2009)

For every positive integer m there is $t_m^+ < \omega$ such that for any cardinal κ which is measurable after generically adding λ many Cohen subsets of κ where $\lambda \to (\kappa)^{2m}_{2^\kappa}$, the κ-dense linear order $\mathbb{Q}_\kappa$ satisfies

$$\mathbb{Q}_\kappa \to (\langle\mathbb{Q}\rangle_\kappa)^m_{<\omega, t_m^+} \text{ and } \mathbb{Q}_\kappa \not\to (\langle\mathbb{Q}\rangle_\kappa)^m_{<\omega, t_m^+ - 1},$$

where t_m^+ is the cardinality of a finite set of trees.

Furthermore, for $m \geq 3$, $t_m^+ > t_m$: $t_3 = 16 < 20 = t_3^+$; $t_4 = 272 < 776 = t_4^+$; $t_5 = 7936 < 151,184 = t_5^+$.

Joint work with Džamonja and Mitchell

A key tool:

Theorem (Saharon Shelah, 1988)

Suppose that $m < \omega$ and κ is a cardinal which is measurable in the generic extension obtained by adding λ Cohen subsets of κ where $\lambda \to (\kappa)^{2m}_{2^\kappa}$. Then for any coloring d of the m-element antichains of ${}^{\kappa>}2$ into $\sigma < \kappa$ colors, and any well-ordering $\prec$ of the levels of ${}^{\kappa>}2$, there is a strong embedding $e : {}^{\kappa>}2 \to {}^{\kappa>}2$ and a dense set of elements w such that

- $e(s) \prec e(t)$ *for all* $s \prec t$ *from* $Cone(w)$, *and*
- $d(e[a]) = d(e[b])$ *for all* $\prec$*-similar* m*-element antichains* a *and* b *of* $Cone(w)$.

Work with Džamonja and Mitchell

Based on work by Hajnal and Komjáth, the positive partition relation for $\mathbb{Q}_\kappa$ does not follow from any large cardinal hypothesis on κ.

Theorem (András Hajnal and Péter Komjáth)

There is a forcing of size $\aleph_1$ which adds an order type θ of size $\aleph_1$ with the property that $\psi \nrightarrow [\theta]^2_{\omega_1}$ for every order type ψ, regardless of its size.

Work with Džamonja and Mitchell

Our proof uses the theorem of Shelah quoted above to show that all the types we say occur do occur in every large set.

Does $\mathbb{Q}_\kappa \nrightarrow (\mathbb{Q})^m_{<\kappa, t^+_{m-1}}$ hold for sufficiently large $m < \omega$ for some regular cardinal which is not measureable after generically adding λ Cohen subsets of κ where $\lambda \rightarrow (\kappa)^{2m}_{2^\kappa}$?

Bibliography

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Thank you!