Partition Relation Perspectives

Jean A. Larson

Menachem Magidor 70th Birthday Conference
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Menachem’s doctoral work

In 1973, Menachem Magidor received his PhD under the direction of Azriel Levy at the Hebrew University of Jerusalem.
First memory of Menachem ~ 1973
Hebrew University of Jerusalem

Menachem spent most of his career at the Hebrew University of Jerusalem.

- He was promoted to Associate Professor in 1978
- He was promoted to Professor in 1982
- He served as president of the university 1997-2009
Menachem’s students (Math Genealogy Project)

- Charly Bitton, Hebrew University, 1998
- Moti Gitik, Hebrew University, 1980 (A. Levy + MM)
- Hiam Judah, Hebrew University, 1987
- Yechiel Kimchi, Hebrew University, 1987
- Gregory Lafitte, Ecole Normal Superieure de Lyon, 2002
- Amir Leshem, Hebrew University, 1998
- Ami Litman, Hebrew University, 1981
- Miri Segal, Hebrew University, 1997
- Anna Sfard, Hebrew University, 1989
- Huiling Zhu, National University of Singapore, 2012
Some co-authors (MathSciNet)

Uri Abraham
Joan Bagaria
Maxim Burke
Paul Erdős
Matthew Foreman
Fred Galvin
Thomash Jech
Richard Ketchersid
Daniel Lehmann
Jerome Malitz
Karel Prikry
Hiroshi Sakai
Saharon Shelah
Jonathan Stav

Ron Aharoni
Shai Ben-David
Brent Cody
Ilijas Farah
Sy-David Friedman
Moti Gitik
Istvan Juhász
Sarat Kraus
Amir Leshem
Willam Mitchell
John Rosenthal
Ralf Schindler
Richard Shore
Jouko Vaananen

Arthur Apter
Rachel Eliyahu-Zohary
James Cummings
Qi Feng
László Fuchs
Yuri Gurevic
Akihiro Kanamori
Paul Larson
Jean-Pierre Levinski
Gadi Moran
Matayahu Rubin
Karl Schlechta
Gabriel Srour
W. Hugh Woodin
Thanks for all the conferences you worked on
The conversation shifts to mathematics
Balanced Baumgartner-Hajnal-Todorcevic (BHT) Theorem, 1993

For every regular uncountable \( \kappa, \xi < \kappa, \) and \( k < \omega, \)

\[(2^\kappa)^+ \to (\kappa + \xi)_k^2.\]

This theorem has a very nice proof using ideals that come from chains of elementary submodels.
Ari Brodsky’s Balanced BHT Theorem for Trees, 2014

For every infinite regular cardinal $\kappa$, $\xi$ with $2^{|\xi|} < \kappa$, and $k < \omega$, non-$\left(2^{<\kappa}\right)$-special tree $\rightarrow (\kappa + \xi)^2_k$.

Brodsky developed a theory of stationary subsets of trees in order to prove this theorem and also used ideals that come from chains of elementary submodels.
Theorem (Paul Erdős and Richard Rado, 1956)

The Continuum Hypothesis implies \((\aleph_1 \omega) \not\rightarrow (\aleph_1 \omega)^{1,1}\)

Here \((\theta \omega) \not\rightarrow (\theta \omega)^{1,1}\) means for every function \(c : \theta \times \omega \rightarrow 2\)
there are \(A \subseteq \theta\) and \(B \subseteq \omega\) such that \(|A| = \theta\), \(B\) is infinite and \(c \upharpoonright (A \times B)\) is constant.
Theorem (Shimon Garti and Saharon Shelah, 2014)

It is consistent with ZFC, that all cardinals \( \theta \) with \( \aleph_1 \leq \theta \leq 2^\omega \),

\[
\left( \begin{array}{c} \theta \\ \omega \end{array} \right) \rightarrow \left( \begin{array}{c} \theta \\ \omega \end{array} \right)^{1,1}_2
\]

Earlier they had shown the partition relation was true for \( \theta < s \) (splitting number) and here show it is true for \( r < \theta \) (reaping number).

Their model has \( \aleph_1 = r = \leq s = \aleph_2 = c \).

Question: is it consistent that \( r < s \) and \( c > \aleph_2 \)?
Topological reduction of colors

Given countable ordinals $\alpha$ and $\beta$, Claribet Piña has investigated the minimum number of colors $m$ needed, so that if pairs from $\alpha$ are colored with finitely many colors (at least 2), then there is a topological copy of $\beta$ whose pairs realize at most $m$ colors.

This may be expressed compactly as $(\forall \ell > 1) \left[ \alpha \to_{top} (\beta)^2_{\ell,m} \right]$.

**Theorem (Piña 2015)**

*For all $\ell > 1$,*

\[ \omega^\omega \to_{top} (\omega^2 + 1)^2_{\ell,6}, \]

*but*

\[ \omega^\omega \not\to_{top} (\omega^2 + 1)^2_{9,5}. \]
Topological Pigeonhole Principle

Building on the ordinal pigeonhold principle of Eric Milner and Richard Rado which calculates \( P^{\text{ord}}(\alpha_i) \), and work of Bill Weiss, of Prikry and Solovay, and of Shelah, Jacob Hilton constructed a pigeonhole principle for ordinal topological spaces. Here is the theorem he describes as his main breakthrough.

**Theorem (Jacob Hilton, JSL, to appear)**

Suppose \( 0 < \alpha_1, \alpha_2, \ldots, \alpha_k < \omega_1 \).

1. \( p^{\text{top}}(\omega^{\alpha_1} + 1, \omega^{\alpha_2} + 1, \ldots, \omega^{\alpha_k} + 1) = \omega^{\alpha_1 \# \alpha_2 \# \ldots \# \alpha_k} + 1 \).

2. \( p^{\text{top}}(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_k}) = \omega^{P^{\text{ord}}(\alpha_1, \alpha_2, \ldots, \alpha_k)} \).

If \( \alpha = \omega^{a_0} \cdot m_0 + \cdots + \omega^{a_p} \cdot m_p \) and \( \beta = \omega^{a_0} \cdot n_0 + \cdots + \omega^{a_p} \cdot n_p \) where \( a_0 > a_1 > \cdots > a_p \) and \( m_0, m_1, \ldots m_p, n_0, n_1, \ldots n_p < \omega \), then the Hessenberg sum (natural sum) is defined by

\[ \alpha \# \beta = \omega^{a_0} \cdot (m_0 + n_0) + \cdots + \omega^{a_p} \cdot (m_p + n_p) \]
In their search for Ramsey numbers of ordinal topological spaces, Andres Caicedo and Jacob Hilton looked at two kinds of goals: $R^{top}$ when the goals were subspaces homeomorphic to given ordinal spaces and $R^{cl}$ when the goals that were subspaces both order-isomorphic to given ordinal spaces and closed in their own supremum.

Caicedo and Hilton, building on Hilton’s work, proved an analogous closed pigeonhole principle, $P^{cl}(\alpha_i)_{i<\kappa}$. 
They write $R^{\text{top}}(\alpha_i)_{i<\kappa}$ for the topological Ramsey number, namely the least ordinal $\beta$ such that

$$\beta \rightarrow \text{top} (\alpha_i)^2_{i<\kappa}$$

Similarly, they write $R^{\text{cl}}(\alpha_i)_{i<\kappa}$ for the closed Ramsey number, namely the least ordinal $\beta$ such that

$$\beta \rightarrow \text{cl} (\alpha_i)^2_{i<\kappa}$$
Topological Ramsey numbers

Paul Erdős and Eric Milner proved the following theorem for ordinal partition relations:

**Theorem (Erdős, Milner)**

If $\omega^\alpha \rightarrow (\omega^{1+\beta}, k)^2$, then $\omega^{\alpha+\beta} \rightarrow (\omega^{1+\beta}, 2k)^2$.

Andres Caicedo and Jacob Hilton found a counterpart which they call the Weak Topological Erdős-Milner Theorem.

**Theorem (Caicedo, Hilton)**

Let $\alpha$ and $\beta$ be countable non-zero ordinals, and let $k > 1$ be a positive integer. If $\omega^\alpha \rightarrow^{top} (\omega^\beta, k)^2$, then $\omega^{\omega^\alpha \cdot \beta} \rightarrow^{top} (\omega^\beta, k + 1)^2$. 
Topological Ramsey numbers

In a corollary to their main theorem, Caicedo and Hilton prove these Ramsey numbers coincide in certain cases and give upper bounds for them.

**Theorem (Caicedo, Hilton)**

*Suppose $0 < \alpha < \omega_1$ and $k, m$ and $n$ are positive integers.*

1. $R^{\text{top}}(\alpha, k)$ is countable.
2. $R^{\text{top}}(\omega^\alpha, k + 1) = R^{\text{cl}}(\omega^\alpha, k + 1) \leq \omega^{\omega^\alpha \cdot k}$.
3. $R^{\text{top}}(\omega^{\omega^\alpha + 1}, k + 1) = R^{\text{cl}}(\omega^{\omega^\alpha + 1}, k + 1)$ and the common value is bounded by $
   \omega^{\omega^\alpha \cdot k + 1}$ if $\alpha$ is infinite, and bounded by $
   \omega^{\omega^{(n+1) \cdot k - 1} + 1}$ if $\alpha = n$ is finite.
Topological Ramsey Theory

Question: is there a topological partition ordinal $\alpha > \omega$?
That is, is there $\alpha > \omega$ for which $\alpha \rightarrow_{top} (\alpha, 3)^2$?

Question: is it possible to reduce the computation of $R^{top}(\alpha, k)$ to finite combinatorial problems, even for $\alpha < \omega^2$?
Halpern-Läuchli Theorem

Copies of illustrations of a $k$-dense subset and a $k$-$x$-dense subset from *Ramsey Theory* by Todorcevic:

A $k$-dense matrix ($k$-$\vec{x}$-dense matrix) for $\prod_{i<d} T_i$ is $\prod_{i<d} X_i$ where each $X_i$ is a $k$-dense matrix ($k$-$x_i$-dense matrix).
Halpern-Läuchli Theorem

(Asymmetric Version of the Halpern-Läuchli Theorem)

For every coloring of the finite product of finitely branching trees, $\prod_{i<d} T_i = K_0 \cup K_1$, either

- $K_0$ contains a $k$-dense matrix for every integer $k$, or
- there is some $\vec{x} \in \prod_{i<d} T_i$ so that $K_1$ contains a $k-\vec{x}$-dense matrix for every integer $k$. 
Laver’s Conjecture:

For every (level set) coloring of the infinite product of downwards closed perfect \( \prod_{i<\omega} T_i = K_0 \cup K_1 \), there is some \( j < 2 \) and some \( \vec{x} \in \prod_{i<\omega} T_i \) such that for each \( k < \omega \), \( K_j \) contains a \( k-\vec{x} \)-dense (level set) matrix.
Theorem (Denis Devlin, 1979)

\[ \mathbb{Q} \rightarrow (\mathbb{Q})^n_{<\omega,t_n} \text{ and } \mathbb{Q} \rightarrow (\mathbb{Q})^n_{<\omega,t_n-1}, \]

where \( t_n \) is the \( n \)th tangent number.

For regular cardinals \( \kappa \), there is a natural order \( \leq_Q \) extending lexicographic order on \( \kappa^+ > 2 \) such that \( \mathbb{Q}_\kappa = \langle \kappa^+ > 2, \leq_Q \rangle \) is a \( \kappa \)-dense linear order.
Joint work with Džamonja and Mitchell

Theorem (Džamonja, Larson, Mitchell, 2009)

For every positive integer $m$ there is $t^+_m < \omega$ such that for any cardinal $\kappa$ which is measurable after generically adding $\lambda$ many Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)^{2m}_{2\kappa}$, the $\kappa$-dense linear order $Q_\kappa$ satisfies

$$Q_\kappa \rightarrow (Q_\kappa)^m_{<\omega,t^+_m} \text{ and } Q_\kappa \not\rightarrow (Q_\kappa)^m_{<\omega,t^+_m-1},$$

where $t^+_m$ is the cardinality of a finite set of trees.

Furthermore, for $m \geq 3$, $t^+_m > t_m$: $t_3 = 16 < 20 = t^+_3$; $t_4 = 272 < 776 = t^+_4$; $t_5 = 7936 < 151,184 = t^+_5$. 
Joint work with Džamonja and Mitchell

A key tool:

**Theorem (Saharon Shelah, 1988)**

Suppose that $m < \omega$ and $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)^{2m}_{2^\kappa}$. Then for any coloring $d$ of the $m$-element antichains of $\kappa > 2$ into $\sigma < \kappa$ colors, and any well-ordering $\prec$ of the levels of $\kappa > 2$, there is a strong embedding $e : \kappa > 2 \rightarrow \kappa > 2$ and a dense set of elements $w$ such that

- $e(s) \prec e(t)$ for all $s \prec t$ from $\text{Cone}(w)$, and
- $d(e[a]) = d(e[b])$ for all $\prec$-similar $m$-element antichains $a$ and $b$ of $\text{Cone}(w)$. 
Based on work by Hajnal and Komjáth, the positive partition relation for $\mathcal{Q}_\kappa$ does not follow from any large cardinal hypothesis on $\kappa$.

Theorem (András Hajnal and Péter Komjáth)

*There is a forcing of size $\aleph_1$ which adds an order type $\theta$ of size $\aleph_1$ with the property that $\psi \not\rightarrow [\theta]^{\aleph_1}_2$ for every order type $\psi$, regardless of its size.*
Our proof uses the theorem of Shelah quoted above to show that all the types we say occur do occur in every large set.

Does $\mathbb{Q}_\kappa \leftrightarrow (\mathbb{Q})^{m}_{<\kappa,t_{m-1}^+}$ hold for sufficiently large $m < \omega$ for some regular cardinal which is not measureable after generically adding $\lambda$ Cohen subsets of $\kappa$ where $\lambda \rightarrow (\kappa)^{2^m}_{2^\kappa}$?
Bibliography

Thank you!