A Borel amalgamation property.

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joint work with Samuel Coskey, Boise State University
Overview

1. Borel reducibility
2. Amalgamation properties
3. The conjugacy problem for automorphism groups of Fraïssé limits
4. A Borel amalgamation property
   - Partial orders
   - Tournaments
5. Katětov functors
A standard Borel space $X$ is the Borel space of a Polish space.

Given equivalence relations $E$ on $X$ and $F$ on $Y$, then a Borel reduction from $E$ to $F$ is a Borel map $f : X \to Y$ satisfying

$$x E x' \iff f(x) F f(x').$$

In this case, we say $E$ is Borel reducible to $F$, and write $E \leq_B F$. 

There are equivalence relations which are universal for the class of isomorphism relations on classes of countable structures. We call such a relation Borel complete. Examples include the isomorphism relations on countable graphs, on countable linear orders, on countable tournaments, and on countable partial orders.

If $E$ is Borel complete, and if $E \leq_B F$ where $F$ is Borel reducible to an isomorphism relation on a class of countable structures, then $F$ is also Borel complete.
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Fraïssé’s Theorem

Let \( \mathcal{K} \) be a class of finite structures which is closed under substructures, and assume for simplicity that the structures are in a finite, relational language. Let \( \mathcal{K}_\omega \) be the class of countable structures, all of whose finite substructures lie in \( \mathcal{K} \). Fraïssé’s Theorem tells us that if \( \mathcal{K} \) satisfies the amalgamation property:

**Amalgamation Property**

For all \( A, B_1, B_2 \in \mathcal{K} \) such that \( f_1 : A \to B_1 \) and \( f_2 : A \to B_2 \) are embeddings, there exists \( D \in \mathcal{K} \) and embeddings \( g_1 : B_1 \to D \) and \( g_2 : B_2 \to D \) satisfying \( g_1 \circ f_1 = g_2 \circ f_2 \).

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then there is a homogeneous structure in $\mathcal{K}_\omega$ (called the Fraïssé limit) whose finite substructures are exactly $\mathcal{K}$, and that this class is unique up to isomorphism. Thus there is a countable homogeneous graph, directed graph, linear order ($\mathbb{Q}$), tournament, partial order, etc. Homogeneous means that every finite partial automorphism extends to a full automorphism.
More amalgamation properties

Let $\mathcal{K}$ be a class of finite relational structures.

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### Free Amalgamation Property (FAP)

$(SAP) \ldots$ and there are no nontrivial relations between the sets $f_1(B_1) \setminus f_1 \circ g_1(A)$ and $f_2(B_2) \setminus f_2 \circ g_2(A)$. 
Automorphism groups of Fraïssé limits

There has been some analysis of the automorphism groups of Fraïssé limits:

Theorem ([C-E-Schneider, 2010])
Let \( \Gamma \) denote the Fraïssé limit for the class of undirected graphs. The isomorphism relation on countable graphs is Borel reducible to the conjugacy relation on \( \text{Aut}(\Gamma) \). Hence this conjugacy relation is Borel complete.

Theorem ([Bilge-Melleray, 2013])
Let \( F \) be a class of finite structures with the FAP, and let \( M \) be the Fraïssé limit. Then the isomorphism relation on \( F^\omega \) is Borel reducible to the conjugacy relation on \( \text{Aut}(M) \).

But the conclusion of this theorem holds also for some amalgamation classes without the FAP.

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A Borel amalgamation property

Given a structure $A \in \mathcal{K}_\omega$, a quantifier-free type $\tau(x, \bar{a})$ with finitely many parameters from $A$ is said to be an admissible finitary type over $A$ (with respect to $\mathcal{K}_\omega$) if there exists $B \in \mathcal{K}_\omega$ such that $A$ is a substructure of $B$ and $B$ contains a witness for $\tau$.
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**Definition**

We say that $\mathcal{K}_\omega$ has the **Borel amalgamation property (BAP)** if there is a Borel assignment $E : \mathcal{K}_\omega \to \mathcal{K}_\omega$ such that $E(A)$ contains $A$ and $E(A)$ contains witnesses for all admissible finite types over $A$. 

Note that $\text{FAP} = \Rightarrow \text{BAP} = \Rightarrow \text{SAP}$. The first implication is not reversible, since for example we will shortly check that the classes of countable partial orders and countable tournaments have the BAP but not the FAP. We conjecture that the second implication is also not reversible.
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In order to establish a generalization of Bilge-Melleray, we actually require more than just the BAP. For lack of a better name, let us say that $K_\omega$ has the **Automorphic BAP (ABAP)** if it has the BAP plus the following properties:

(a) For any $A \in K_\omega$ and automorphism $\varphi: A \to A$, there is an automorphism $\tilde{\varphi}: E(A) \to E(A)$ extending $\varphi$ with no fixed points in $E(A) \setminus A$. Moreover $\tilde{\varphi}$ can be selected from the parameters $A$ and $\varphi$ in a Borel fashion.

(b) For any $A, A' \in K_\omega$ and isomorphism $\alpha: A \to A'$, there is an extension $\hat{\alpha}: E(A) \to E(A')$ with the following property: given $A, A' \in K_\omega$ and automorphisms $\varphi, \varphi'$ of $A, A'$ respectively, if $\alpha$ conjugates $\varphi$ to $\varphi'$ then $\hat{\alpha}$ conjugates $\tilde{\varphi}$ to $\tilde{\varphi}'$. 
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Moreover any isomorphism $\alpha: P \to P'$ naturally extends to $\hat{\alpha}: E(P) \to E(P')$ by mapping the witnesses of a given admissible finitary type over $P$ to the witnesses of the corresponding type over $P'$. 
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Of course the FAP fails because it is necessary to close the order relation under transitivity.
ABAP does what I said it does

Theorem

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If $\alpha : A \to A'$ is an isomorphism, then use condition (b) repeatedly to obtain an isomorphism $\alpha_\infty : A_\infty \to A'_\infty$. 
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On the other hand, note that for any $A \in K_\omega$, condition (a) together with the definition of $\varphi_A$ implies that the set of fixed points of $\varphi_A$ is precisely $A_0 = A$. Thus if $A, A' \in K_\omega$ and $\varphi_A$ is conjugate to $\varphi_{A'}$, then the conjugator witnesses that $A$ is isomorphic to $A'$.
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Nevertheless tournaments do have the ABAP.
POs were straightforward, but what about tournaments?

It is not surprising that the class of partial orders has the ABAP. We can amalgamate partial orders ‘essentially freely’.

But to amalgamate tournaments, we must ‘make choices’.

Nevertheless tournaments do have the ABAP.

The following construction is adapted from [Kubiš-Mašulović, 2015]:
The class of countable tournaments also satisfy the ABAP:

Given a countable tournament $T$, we let the vertex set of $E(T)$ consist of vertices of $T$ together with elements with names $(s,n)$ where $s$ is a finite sequence from $T$ and $n \in \mathbb{Z}$. For an old vertex $v \in T$ and a new vertex $(s,n)$, set $v \rightarrow (s,n)$ if $v$ appears in $s$ and $(s,n) \rightarrow v$ otherwise.

The edges between the new vertices are defined lexicographically: if $|s| < |s'|$ we let $(s,n) \rightarrow (s',n')$; if $|s| = |s'|$, $i$ is the first coordinate where they differ, and $s_i \rightarrow s_i'$, we let $(s,n) \rightarrow (s',n')$. Finally we let $(s,n) \rightarrow (s,n+k)$ for all $k > 0$.

Now any automorphism $\varphi$ of $T$ can be extended to $E(T)$ by letting $\tilde{\varphi}(s,n) = (\varphi(s),n+1)$, where $\varphi(s)$ denotes the coordinatewise application of $\varphi$.

Moreover any isomorphism $\alpha: T \rightarrow T'$ naturally extends to $E(T) \rightarrow E(T')$ by mapping $\hat{\alpha}(s,n) = (\alpha(s),n)$.
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An interesting example

This example is due to [Grebik, 2016]:

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Let \( \mathcal{L} \) be the class of finite linear orders with a coloring on the pairs of vertices (using \( \omega \)-many colors) so that there are no monochromatic triangles.
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But $\mathcal{L}_\omega$ does not seem to have the ABAP.

And $\mathcal{L}$ seems to satisfy the conclusion to [Bilge-Melleray, 2013] ... but only because the conjugacy relation *should* be Borel complete!
Questions

Conjecture

SAP does not imply ABAP. (Grebik’s example)
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Question
What is the exact relationship between the ABAP and the existence of a Katětov functor?
Katětov functors

**Definition**

Let $\mathcal{K}$ be a Fraïssé class and $\mathcal{K}_\omega$ the corresponding class of countable structures whose age is contained in $\mathcal{K}$. A Katětov functor consists of

- a map $E : \mathcal{K}_\omega \to \mathcal{K}_\omega$ such that $A \subset E(A)$
- a map $\hat{\cdot}$ which assigns to each embedding $f : A \to B$ an embedding $\hat{f} : E(A) \to E(B)$ such that $f \subset \hat{f}$

such that

**Functoriality** $\hat{f} \circ \hat{g} = \hat{(f \circ g)}$ and $\hat{id}_A = id_{E(A)}$

**One-point extensions** for all $A \in \mathcal{K}_\omega$ and all admissible finite types $\tau$ over $A$, there exists a witness for $\tau$ in $E(A)$.

The existence of a Katětov functor implies that $\mathcal{K}$ satisfies SAP.
References

Samuel Coskey, Paul Ellis, Scott Schneider
The conjugacy problem for the automorphism group of the random graph

Dogan Bilge and Julien Melleray.
Elements of finite order in automorphism groups of homogeneous structures.

Wiesław Kubiś and Dragan Mašulović.
Katětov functors.

Jan Grebík
An example of a Fraïssé class without a Katětov functor

Samuel Coskey, Paul Ellis
The conjugacy problem for automorphism groups of homogeneous digraphs