Partition Properties for Non-Ordinal Sets Under the Axiom of Determinacy

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Finite Partition Properties

Definition
For $\kappa$ a cardinal and $n \in \omega$,
$$[\kappa]^n = \{ (\alpha_1, \cdots, \alpha_n) \in \kappa^n : \alpha_1 < \cdots < \alpha_n \}.$$ We also set
$$[\kappa]^<\omega = \bigcup_{n \in \omega} [\kappa]^n.$$

Definition
Let $\kappa, \lambda, \delta$ be cardinals and $n \in \omega$.

- $[\kappa]^<\omega \rightarrow [\kappa]^<\omega$ means: for every $f : [\kappa]^<\omega \rightarrow \lambda$, there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and $|f([H]^n)| \leq \delta$ for all $n$.

- $[\kappa]^<\omega \rightarrow [\kappa]^<_\lambda$ means: $(\kappa)^<_\mu \rightarrow (\kappa)^<_\delta$ for all $\mu < \lambda$.

- $\kappa$ is Ramsey iff $[\kappa]^<_2 \rightarrow [\kappa]^<_1$.

- $\kappa$ is Rowbottom iff $[\kappa]^<_\kappa \rightarrow [\kappa]^<_\omega$. 
Finite Partition Properties

Definition

$\kappa$ is Jónsson iff for every $f : [\kappa]^{<\omega} \to \kappa$, there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and $f[H]^{<\omega} \neq \kappa$.

Remark

In ZFC, Ramsey implies Rowbottom and Jónsson, and both Rowbottom and Jónsson imply the existence of $0^\#$ and thus that $V \neq L$. 
Some Determinacy Notions

Definition
Recall that under the axiom of determinacy (AD), \( \mathbb{R} \) cannot be well-ordered. We define \( \Theta \) to be least cardinal that \( \mathbb{R} \) does not surject onto.

Definition
Recall that \( L(\mathbb{R}) \) is the minimal universe of ZF which contains \( \mathbb{R} \). Under large cardinal hypotheses, \( L(\mathbb{R}) \) is a model of AD, and its theory is absolute for very complex statements.
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Remark
It has been shown that under AD, ordinary cardinals have large cardinal properties in \( L(\mathbb{R}) \). For instance, \( \omega_1 \) is a measurable cardinal.
Finite Partition Properties Under AD

In 2015, S. Jackson, R. Ketchersid, F. Schlutzenberg, and W.H. Woodin [3] proved the following:

**Theorem (AD + V = L(\mathbb{R}), J/K/S/W)**

Let $\kappa < \Theta$ be an uncountable cardinal. Then:

1. If $\text{cf}(\kappa) = \omega$, then $\kappa$ is Rowbottom.

2. $\kappa$ is Jónsson. In fact, if $\lambda$ is a cardinal between $\omega_1$ and $\kappa$, and $f : [\kappa]^{<\omega} \to \lambda$, then there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and

\[ |\lambda - f([H]^{<\omega})| = \lambda. \]
Finite Partition Properties Under AD

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2. \( \kappa \) is Jónsson. In fact, if \( \lambda \) is a cardinal between \( \omega_1 \) and \( \kappa \), and \( f : [\kappa]^{<\omega} \to \lambda \), then there is an \( H \subseteq \kappa \) so that \( |H| = \kappa \) and

\[
|\lambda - f([H]^{<\omega})| = \lambda.
\]

In this paper, they asked whether or not there were non-ordinal Jónsson cardinals. In particular, is \( \bR \) Jónsson?
Reframing the Question

Definition
For any set $A$, $[A]^n = \{s \subseteq X : |s| = n\}$ and $[A]^{<\omega} = \bigcup_{n \in \omega} [A]^n$.

Definition
Let $A$ and $B$ be infinite sets.

- $(A, B)$ is **Ramsey** iff for any $f : [A]^{<\omega} \rightarrow B$, there is an $X \subseteq A$ so that $|X| = |A|$ and $f$ is constant on each $[X]^n$.

- $(A, B)$ is **Rowbottom** iff for any $f : [A]^{<\omega} \rightarrow B$, there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}]$ is countable.

- $(A, B)$ is a **strong Jónsson pair** iff for any $f : [A]^{<\omega} \rightarrow B$, there is an $X \subseteq A$ so that $|X| = |A|$ and

\[|B - f[[X]^{<\omega}]| = |B|.\]
Tools From Descriptive Set Theory

We use the following repeatedly.

**Lemma (Fusion Lemma)**

*For each* \( s \in 2^{<\omega} \) *let* \( P_s \) *be a perfect set so that*

1. \( \lim_{|s| \to \infty} \text{diam}(P_s) = 0 \), and
2. for all \( s \in 2^{<\omega} \), \( P_{s0} \cap P_{s1} = \emptyset \) and \( P_{s0}, P_{s1} \subseteq P_s \).

*Then the fusion* \( P = \bigcup_{f \in 2^\omega} \bigcap_{n \in \omega} P_{f|n} \) \( \langle P_s : s \in 2^{<\omega} \rangle \) *is a perfect set.*

**Theorem (Mycielski)**

*Suppose* \( C_n \subseteq (2^\omega)^n \) *are comeager for all* \( n \in \omega \). *Then there is a perfect set* \( P \subseteq 2^\omega \) *so that* \( [P]^n \subseteq C_n \) *for all* \( n \).
$\mathbb{R}$ is Strongly Jónsson

Theorem (AD, Holshouser/Jackson)

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**Theorem (AD, Holshouser/Jackson)**

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**Proof.**

- We can break \( f \) into component functions, \( f_n \).
- Find comeager sets on which the \( f_n \) are continuous.
- Use the result of Mycielski[4] to thread a perfect set through the comeager sets.
- Use continuity and the fusion lemma to inductively thin out the range of the \( f_n \).
Jónsson Properties for $\mathbb{R}$

Jónsson Properties for General Sets

Jónsson Properties for $\mathbb{R}/E_0$

Domain $f_1$ Range

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\[\mathbb{R}\] and Cardinals

Proposition (AD)

If \(\kappa < \Theta\) is an uncountable cardinal, then \((\mathbb{R}, \kappa)\) and \((\kappa, \mathbb{R})\) are Rowbottom.

Proposition (AD + V = L(\(\mathbb{R}\)), Holshouser/Jackson)

Let \(\kappa, \lambda < \Theta\) be uncountable cardinals. Suppose

\[A, B \in \{\kappa, \lambda, \mathbb{R}, \kappa \cup \mathbb{R}, \kappa \times \mathbb{R}, \lambda \cup \mathbb{R}, \lambda \times \mathbb{R}\}\]

Then \((A, B)\) is a strong Jónsson pair.
\( \mathbb{R} \) and Cardinals

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What about other non-ordinal sets?
Describing More General Sets

Suppose $X \in L_\Theta(\mathbb{R})$. Then there is a surjection $F : \mathbb{R} \to X$. We can define an equivalence relation $E$ on $\mathbb{R}$ by

$$xEy \iff F(x) = F(y).$$

Note that $X$ is in bijection with $\mathbb{R}/E$. So we only need to consider quotients of $\mathbb{R}$. 
Jónsson Properties for General Quotients

There is a (possibly not unique) decomposition of $\mathbb{R}/E$ into a well-ordered component and another component which $\mathbb{R}$ surjects onto and injects into [2]. Call the surjection $\phi^X$ and the injection $\phi_X$. Either of these components could be empty.
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**Theorem (AD + $V = L(\mathbb{R})$, Holshouser/Jackson)**

Suppose that $X \in L_\Theta(\mathbb{R})$ is in bijection with $\kappa \cup A$, where $\kappa$ is an uncountable cardinal and $\mathbb{R}$ maps onto and into $A$. Similarly, suppose $Y \in L_\Theta(\mathbb{R})$ is in bijection with $\lambda \cup B$. Let $f : [\kappa \cup A]^\omega \to \lambda \cup B$. Then there are perfect $P, Q \subseteq \mathbb{R}$ and there is an $H \subseteq \kappa$ with $|H| = \kappa$ so that

$$|\lambda - f[[H \cup \phi^A[P]]^\omega]| = \lambda \text{ and } f[[H \cup \phi^A[P]]^\omega] \cap \phi_B[Q] = \emptyset.$$
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$$|\lambda - f[[H \cup \phi^A[P]]^{<\omega}]| = \lambda \text{ and } f[[H \cup \phi^A[P]]^{<\omega}] \cap \phi_B[Q] = \emptyset.$$  

This is unsatisfactory as this result does not give us bijections.
Background for $E_0$

Recall the following:

**Definition**

Let $x, y \in 2^\omega$. Then $xE_0y$ iff $(\exists N)(\forall n \geq N)[x(n) = y(n)]$.

Note that $2^\omega/E_0$ has no definable linear ordering and $E_0$ has no definable transversal.
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Let $x, y \in 2^\omega$. Then $xE_0y$ iff $(\exists N)(\forall n \geq N)[x(n) = y(n)]$.

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The following is a corollary of the Glimm-Effros Dichotomy [1]:

**Corollary (AD)**

Suppose $H \subseteq 2^\omega/E_0$. Then $H$ satisfies exactly one of the following:

- $H$ is countable,
- $H$ is in bijection with $\mathbb{R}$, or
- $H$ is in bijection with $2^\omega/E_0$. 
Mycielski for $E_0$

**Definition**
$A \subseteq 2^\omega$ has **power $E_0$** iff $A$ is $E_0$-saturated and $A/E_0$ is in bijection with $2^\omega/E_0$.

**Definition**
For $n \in \omega$ and $A \subseteq 2^\omega$, let

$$[A]^n_{E_0} = \{ \bar{x} \in [A]^n : |\{[x_1]_{E_0}, \ldots, [x_n]_{E_0}\}| = n \}$$
Mycielski for $E_0$

**Definition**
$A \subseteq 2^\omega$ has **power** $E_0$ iff $A$ is $E_0$-saturated and $A/E_0$ is in bijection with $2^\omega/E_0$.

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For $n \in \omega$ and $A \subseteq 2^\omega$, let

$$[A]^n_{E_0} = \{ \vec{x} \in [A]^n : |\{[x_1]_{E_0}, \ldots, [x_n]_{E_0}\}| = n \}$$

We were able to prove the following Mycielski style result.

**Theorem (Holshouser/Jackson)**

Suppose that $C_n \subseteq (2^\omega)^n$ are comeager and $E_0$-saturated for all $n \in \omega$. Then there is an $A \subseteq 2^\omega$ of power $E_0$ so that $[A]^n_{E_0} \subseteq C_n$ for all $n$. 
$\mathbb{R}/E_0$ is Strongly Jónsson

Theorem (AD, Holshouser/Jackson)

$2^\omega/E_0$ is strongly Jónsson.
\( \mathbb{R}/E_0 \) is Strongly Jónsson

**Theorem (AD, Holshouser/Jackson)**

\( 2^\omega/E_0 \) is strongly Jónsson.

**Proof.**

- We can lift \( f : [2^\omega/E_0]^{<\omega} \rightarrow 2^\omega/E_0 \) to a function \( F : [2^\omega]^{<\omega} \rightarrow 2^\omega \) so that
  \[
  \vec{a}E_0\vec{b} \iff F(\vec{a}) \in f(\{[b_1]_{E_0}, \ldots, [b_n]_{E_0}\}).
  \]

- We can break \( F \) into component functions, \( F_n \).
- Find comeager sets on which the \( F_n \) are continuous.
- Use the Mycielski-style result for \( E_0 \) to thread a power \( E_0 \) set through the comeager sets.
- Use continuity and the techniques of the Mycielski-style result to inductively thin out the range of the \( F_n \).
Combinations

Proposition (AD + V = L(R), Holsouer/Jackson)

Let $\kappa, \lambda < \Theta$ be uncountable cardinals. Suppose

$$A, B \in \{ \kappa, \lambda, \mathbb{R}, 2^\omega / E_0, \kappa \cup \mathbb{R}, \kappa \times \mathbb{R}, \lambda \cup \mathbb{R}, \lambda \times \mathbb{R} \} \cup \{ \kappa \cup 2^\omega / E_0, \kappa \times 2^\omega / E_0, \lambda \cup 2^\omega / E_0, \lambda \times 2^\omega / E_0 \}$$

Then $(A, B)$ is a strong Jónsson pair.
More Finite Partition Properties

Proposition (AD, Holshouser/Jackson)

Let $\kappa < \Theta$ be an uncountable cardinal. Then

- $(2^\omega / E_0, \mathbb{R})$ is Ramsey,
- $(2^\omega / E_0, \kappa)$ is Ramsey, and
- $(\kappa, 2^\omega / E_0)$ is Rowbottom.

Proposition (AD + $V = L(\mathbb{R})$, Holshouser/Jackson)

Suppose $\lambda, \kappa < \Theta$ are uncountable cardinals. Then

- $(\kappa \cup 2^\omega / E_0, \mathbb{R})$ is Rowbottom,
- if $\text{cf}(\kappa) = \omega$ and $\lambda < \kappa$, then $(\kappa \cup 2^\omega / E_0, \lambda)$ is Rowbottom, and
- $(2^\omega / E_0, \kappa \cup \mathbb{R})$ and $(2^\omega / E_0, \kappa \times \mathbb{R})$ are Ramsey.
Further Work

- Can the result be extended to well-ordered unions of hyperfinite quotients of $\mathbb{R}$?
- Do infinite partition properties hold for $2^\omega/E_0$?
- Can we get this Mycielski style result for other equivalence relations?
- Can the full Jónsson result be proved for general equivalence relations?
| Background | Jónsson Properties for $\mathbb{R}$ | Jónsson Properties for General Sets | Jónsson Properties for $\mathbb{R}/E_0$ |

**Thanks For Listening!**
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