Lecture III. The complexity of classification problems in ergodic theory

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Copenhagen; June 2016

The last two decades have seen the emergence of a theory of set theoretic complexity of classification problems in mathematics. In this talk I will survey recent developments concerning the application of this theory to classification problems in ergodic theory.

Measure preserving transformations

Definition

A standard measure space is a measure space (X, μ), where X is a Polish space and μ a non-atomic Borel probability measure on X.

All such spaces are isomorphic to the unit interval with Lebesgue measure.

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- $X = \mathbb{T}$ with the usual measure; $T(z) = az$, where $a \in \mathbb{T}$, i.e., T is a rotation.
- $X = 2^{\mathbb{Z}}, T(x)(n) = x(n-1)$, i.e., the shift transformation.

Definition

A mpt T is ergodic if every T-invariant set has measure 0 or 1.

Any irrational, modulo π, rotation and the shift are ergodic.

The ergodic decomposition theorem shows that every mpt can be canonically decomposed into a (generally continuous) direct sum of ergodic mpt's.

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Classifying measure preserving transformations

Starting with work of von Neumann in the early 1930's, in ergodic theory one is interested in classifying ergodic mpt up to various notions of equivalence. I will consider below two such standard notions.

- Isomorphism or conjugacy: A mpt S on (X, μ) is isomorphic to a mpt T on (Y, ν), in symbols $S \cong T$, if there is an isomorphism φ of (X, μ) to (Y, ν) that sends S to T, i.e., $S = \varphi^{-1} T \varphi$.
- Unitary isomorphism: To each mpt T on (X, μ) we can assign the unitary operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$ given by $U_T(f)(x) = f(T^{-1}(x))$. Then S, T are unitarily isomorphic, in symbols $S \cong^u T$, if U_S, U_T are isomorphic.

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Two classical classification theorems:

- (Halmos-von Neumann) An ergodic mpt has discrete spectrum if U_T has discrete spectrum, i.e., there is a basis consisting of eigenvectors. In this case the eigenvalues are simple and form a (countable) subgroup of $\mathbb{T}$. It turns out that up to isomorphism these are exactly the ergodic rotations in compact metric groups $G : T(g) = ag$, where $a \in G$ is such that $\{a^n : n \in \mathbb{Z}\}$ is dense in G. For such T, let $\Gamma_T \leq \mathbb{T}$ be its group of eigenvalues. Then we have:

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- (Ornstein) Let $Y = \{1, \dots, n\}$, $\bar{p} = (p_1, \cdots, p_n)$ a probability distribution on Y and form the product space $X = Y^{\mathbb{Z}}$ with the product measure μ. Consider the Bernoulli shift $T_{\bar{p}}$ on X. Its entropy is the real number $H(\bar{p}) = -\sum_i p_i \log p_i$. Then we have:

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Classification problems

I will next give an introduction to recent work in descriptive set theory, developed primarily over the last 25 years, concerning a theory of complexity of classification problems in mathematics, and then discuss its implications to the above problems.

Classification problems

A classification problem is given by:

- A collection of objects X.
- An equivalence relation E on X.

A complete classification of X up to E consists of:

- A set of invariants I.
- A map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$.

For this to be of any interest both I, c must be as explicit and concrete as possible.

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Classification problems

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Classification of Bernoulli shifts up to isomorphism (Ornstein).
INVARIANTS: Reals

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Classification of ergodic measure-preserving transformations with discrete spectrum up to isomorphism (Halmos-von Neumann).
INVARIANTS: Countable subsets of $\mathbb{T}$.

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Classification of unitary operators on a separable Hilbert space up to isomorphism (Spectral Theorem).
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Most often the collection of objects we try to classify can be viewed as forming a "nice" space, namely a Polish space, and the equivalence relation E turns out to be *Borel* or *analytic* (as a subset of X^2).

For example, in studying mpt the appropriate space is the Polish group of mpt of a fixed (X, μ), with the so-called weak topology. Isomorphism then corresponds to conjugacy in that group, which is an analytic equivalence relation. Similarly unitary isomorphism is an analytic equivalence relation.

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Equivalence relations and reducibility

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Let $(X, E), (Y, F)$ be equivalence relations. E is (Borel) reducible to F, in symbols

$$E \leq_B F,$$

if there is Borel map $f : X \to Y$ such that

$$x \; E \; y \Leftrightarrow f(x) \; F \; f(y).$$

Intuitive meaning:

- The classification problem represented by E is at most as complicated as that of F.
- F-classes are complete invariants for E.

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(Isomorphism of ergodic discrete spectrum mpt) $\sim_B E_c$,
where E_c is the equivalence relation on $\mathbb{T}^{\mathbb{N}}$ given by

$$(x_n)\ E_c\ (y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$$

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(Isomorphism of unitary operators) $\sim_B$ ME,
where ME is the equivalence relation on the Polish space of probability Borel measures on $\mathbb{T}$ given by

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Borel cardinality theory

The preceding concepts can be also interpreted as the basis of a "definable" or Borel cardinality theory for quotient spaces.

- $E \leq_B F$ means that there is a Borel injection of X/E into Y/F, i.e., X/E has Borel cardinality less than or equal to that of Y/F, in symbols
$$|X/E|_B \leq |Y/F|_B$$
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Types of classification and mpt

An equivalence relation E on X is called concretely classifiable if $E \leq_B (=_Y)$, for some Polish space Y, i.e., there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x) = f(y)$.

Thus isomorphism of Bernoulli shifts is concretely classifiable. However in the 1970's Feldman showed that this fails for arbitrary mpt (in fact even for the so-called K-automorphisms, a more general class of mpt than Bernoulli shifts).

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An equivalence relation is called classifiable by countable structures if it can be Borel reduced to isomorphism of countable structures (of some given type, e.g., groups, graphs, linear orderings, etc.). More precisely, given a countable language L, denote by X_L the space of L-structures with universe $\mathbb{N}$. This is a Polish space. Denote by $\cong$ the equivalence relation of isomorphism in X_L. We say that an equivalence relation is classifiable by countable structures if it is Borel reducible to isomorphism on X_L, for some L.

Such types of classification occur often, for example, in operator algebras, topological dynamics, etc.

Types of classification and mpt

It follows from the Halmos-von Neumann theorem that isomorphism (and unitary isomorphism) of ergodic discrete spectrum mpt is classifiable by countable structures. On the other hand we have:

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This has more recently been strengthened as follows:

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Theorem (K, 2007)

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More recently, Foreman-Rudolph-Weiss also showed the following:

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The isomorphism relation of ergodic mpt is not Borel (it is clearly analytic).

However note that unitary isomorphism of mpt is Borel.

Types of classification and mpt

Last year Foreman and Weiss have actually shown that this complexity appears even on very concrete transformations appearing in smooth dynamics.

Theorem (Foreman-Weiss, 2015)

If M is the torus $\mathbb{T}^2$, then the isomorphism relation of ergodic mpt C^∞ diffeomorphisms of M is not Borel.

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It follows from earlier theorems that

$$(\cong^u) \ <_B \ (\cong),$$

i.e., isomorphism of ergodic mpt is strictly more complicated than unitary isomorphism.

We have now seen that the complexity of unitary isomorphism of ergodic mpt can be calculated exactly and there are very strong lower bounds for isomorphism but its exact complexity is unknown. An obvious upper bound is the universal equivalence relation induced by a Borel action of the automorphism group of the measure space.

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Is isomorphism of ergodic mpt Borel bireducible to the universal equivalence relation induced by a Borel action of the automorphism group of the measure space?

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Actions of countable groups

More generally one also considers in ergodic theory the problem of classifying measure preserving actions of countable (discrete) groups Γ on standard measure spaces. The case $\Gamma = \mathbb{Z}$ corresponds to the case of single transformations. We will now look at this problem from the point of view of the preceding theory.

Actions of countable groups

We will consider again isomorphism (also called conjugacy) and unitary isomorphism of actions. Two actions of the group Γ are isomorphic if there is a measure-preserving isomorphism of the underlying spaces that conjugates the actions. They are unitarily isomorphic if the corresponding unitary representations (the Koopman representations) are isomorphic.

We can form again in a canonical way a Polish space $A(\Gamma, X, \mu)$ of all measure-preserving actions of Γ on (X, μ) and then isomorphism and unitary isomorphism become analytic equivalence relations on this space. We can therefore study their complexity using the concepts introduced earlier.

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Actions of countable groups

Theorem (Foreman - Weiss, Hjorth, 2004)

For any infinite countable group Γ, isomorphism of free, ergodic, measure-preserving actions of Γ is not classifiable by countable structures.

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Actions of countable groups

The result of Foreman-Rudolph-Weiss mentioned earlier shows that for $\Gamma = \mathbb{Z}$ the relation of isomorphism of free, ergodic, measure-preserving actions is not Borel but this is unknown for arbitrary Γ. On the other hand Hjorth-Törnquist, answering an old question of Effros from the 1960's, showed that unitary equivalence of unitary representations of any group Γ is Borel, in fact $\mathbf{\Pi^0_3}$, which implies the following:

Theorem (Hjorth-Törnquist, 2012)

For any countable group Γ, unitary isomorphism of measure-preserving actions of Γ is $\mathbf{\Pi^0_3}$.

Orbit equivalence

There is an additional important concept of equivalence between actions, called orbit equivalence. The study of orbit equivalence is a very active area today that has its origins in the connections between ergodic theory and operator algebras and the pioneering work of Dye.

Definition

Given an action of the group Γ on X we associate to it the orbit equivalence relation E^X_Γ, whose classes are the orbits of the action. Given measure-preserving actions of two groups Γ and Δ on spaces (X, μ) and (Y, ν), resp., we say that they are orbit equivalent if there is an isomorphism of the underlying measure spaces that sends E^X_Γ to E^Y_Δ.

Thus isomorphism clearly implies orbit equivalence but not vice versa.

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Orbit equivalence

Here we have the following classical result.

Theorem (Dye, 1959; Ornstein - Weiss, 1980)

Every two free, ergodic, measure-preserving actions of amenable groups are orbit equivalent.

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Orbit equivalence

The situation for non-amenable groups has taken much longer to untangle. For simplicity, below “action” will mean “free, ergodic, measure-preserving action”. Schmidt, 1981, showed that every non-amenable group which does not have Kazhdan’s property (T) admits at least two non-orbit equivalent actions and Hjorth, 2005, showed that every non-amenable group with property (T) has continuum many non-orbit equivalent actions. So every non-amenable group has at least two non-orbit equivalent actions.

Orbit equivalence

For general non-amenable groups though very little was known about the question of how many non-orbit equivalent actions they might have. For example, until recently only finitely many distinct examples of non-orbit equivalent actions of the free (non-abelian) groups were known. Gaboriau – Popa, 2005, finally showed that the free groups have continuum many non-orbit equivalent actions. In an important extension, Ioana, 2007, showed that every group that contains a free subgroup has continuum many such actions. However there are examples of non-amenable groups that contain no free subgroups (Olshanski).

Orbit equivalence

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Orbit equivalence

This still leaves open the possibility that there may be a concrete classification of actions of some non-amenable groups up to orbit equivalence. However the following has been now proved by combining work of Ioana-K-Tsankov and the work of Epstein.

Theorem (Epstein-Ioana-K-Tsankov, 2009)

Orbit equivalence of free, ergodic, measure preserving actions of any non-amenable group is not classifiable by countable structures.

Thus we have a very strong dichotomy:

- If a group is amenable, it has exactly one action up to orbit equivalence.
- If it non-amenable, then orbit equivalence of its actions is unclassifiable in a strong sense.

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