

# Lecture IV: Descriptive graph combinatorics

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# Introduction

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The object of study is the theory of definable graphs, usually Borel or analytic, on standard Borel spaces (Polish spaces with their Borel structure) and one investigates how combinatorial concepts, such as colorings and matchings, behave under definability constraints, i.e., when they are required to be definable or perhaps well-behaved in the topological or measure theoretic sense.

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Instead of a systematic exposition, which would take too long, I will discuss today a few representative results in this theory that give the flavor of the subject.



# Chromatic numbers

A **coloring** of a graph  $G = (V, E)$  is a map from the set of vertices  $V$  of  $G$  to a set  $C$  (the set of colors) such that adjacent vertices are assigned different colors. The **chromatic number** of the graph  $G$ ,  $\chi(G)$ , is the smallest cardinality of such a  $C$ .

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A graph  $G$  is **bipartite** if the vertices can be split into two disjoint sets  $V = A \sqcup B$  such that that edges only connect vertices between  $A$  and  $B$ . This is equivalent to  $\chi(G) \leq 2$ . It is also equivalent to the non-existence of odd cycles. In particular, every acyclic graph is bipartite.

Suppose now  $G = (V, E)$  is a Borel graph (i.e.,  $V$  is a standard Borel space and  $E$  is a Borel set in  $V^2$ ). A **Borel coloring** of the graph  $G = (V, E)$  is a Borel map from the set of vertices  $V$  of  $G$  to a standard Borel space  $C$  (the set of colors) such that adjacent vertices are assigned different colors. The **Borel chromatic number** of the graph  $G$ ,  $\chi_B(G)$ , is the smallest cardinality of such a  $C$ . It is thus equal to one of

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Given a probability Borel measure  $\mu$  on  $V$ , we similarly define the **measurable chromatic number** of  $G$ ,  $\chi_\mu(G)$ , and if  $V$  is a Polish space we define the **Baire measurable chromatic** number of  $G$ ,  $\chi_{BM}(G)$ .

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iii) (KST) Every Borel graph with bounded degree  $\leq d$  has Borel chromatic number  $\leq d + 1$ . (Conley-K, 2009) There are bounded degree, acyclic Borel graphs whose Borel chromatic number takes any finite value. (Marks, 2015) There are  $d$ -regular, acyclic Borel graphs whose Borel chromatic number takes any value in  $\{1, 2, \dots, d + 1\}$ .

# Borel chromatic numbers of shift graphs

Of special interest are graphs generated by group actions. Let  $(\Gamma, S)$  be a **marked group**, i.e, a group with a finite, symmetric set of generators  $S$ . If  $a$  is a free Borel action of  $\Gamma$  on a standard Borel space  $V$  this gives rise to a Borel graph on  $V$ , the “Cayley graph” of the action, where two vertices  $x, y \in V$  are connected iff a generator  $s \in S$  sends  $x$  to  $y$ .

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Every connected component of this graph is a copy of the Cayley graph of  $(\Gamma, S)$ , so this graph has the same chromatic number as the Cayley graph of the group. However the Borel chromatic number behaves very differently and reflects the complexity of the group and the action.

# Borel chromatic numbers of shift graphs

Consider the shift action of the group  $\Gamma$  on  $[0, 1]^\Gamma$ , restricted to its free part. Denote its “Cayley graph” by  $G_\infty(\Gamma, S)$ . On general grounds this has the highest Borel chromatic number among all free actions of  $\Gamma$  and it is bounded by  $d + 1$ , where  $d = |S|$ .

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Take now the groups  $\mathbb{Z}^n$  and  $\mathbb{F}_n$ , with their usual set of generators  $S$ , which we will not explicitly indicate below. The graphs  $G_\infty(\mathbb{Z}^n), G_\infty(\mathbb{F}_n)$  are both bipartite, so they have chromatic number 2. But we have two contrasting pictures when we look at the Borel chromatic numbers:

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**Theorem (Conley-K, Lyons-Nazarov, 2009)**

$$\chi_B(G_\infty(\mathbb{F}_n)) \rightarrow \infty, \text{ as } n \rightarrow \infty$$

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Theorem (Gao-Jackson-Krohne-Seward, 2015)

$$\chi_B(G_\infty(\mathbb{Z}^n)) = 3$$



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Theorem (Bernshteyn, 2016)

$$\chi_\mu(G_\infty(\mathbb{F}_n)) \leq (1 + o(1)) \frac{2n}{\log 2n}$$

Thus

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By contrast, Conley and B. Miller (2014) have shown that the Baire measurable chromatic number  $\chi_{BM}(G_\infty(\mathbb{F}_n))$  is also equal to 3.

$1, 2, \dots, 2n + 1, \aleph_0$

Let  $f_1, f_2, \dots, f_n$  be Borel functions on a standard Borel space  $V$ . Consider the Borel graph  $G_{f_1, f_2, \dots, f_n}$  with vertex set  $V$ , where  $x, y \in V$  are connected by an edge iff there is  $i \leq n$  such that  $f_i(x) = y$  or  $f_i(y) = x$ . (Equivalently this is the undirected version of a directed Borel graph of out-degree  $\leq n$ .) What is the Borel chromatic number of this graph?

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**Problem (K-Solecki-Todorcevic)**

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#### Example (KST)

Consider the space  $V$  of all increasing sequences of natural numbers and let  $s$  be the shift map. Then  $\chi_B(G_s) = \aleph_0$ .

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### Theorem

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- *The answer is positive for every  $n$  if the functions are commuting and fixed-point free (Palamourdas, 2012).*
- *For each  $n \geq 3$ , the Borel chromatic number is in the set:  $\{1, 2, \dots, \frac{1}{2}(n+1)(n+2) - 2, \aleph_0\}$  (Palamourdas, 2012; Meehan, 2015).*

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Finally, B. Miller and Palamourdas showed that if one is willing to throw away a meager set or a null set (for any Borel measure), then the Borel chromatic numbers of these graphs are finite.

# Vizing's Theorem

Given a graph  $G$ , its **edge chromatic number**, in symbols,  $\chi'(G)$ , is the smallest number of colors that we can use to color the edges of the graph so that adjacent edges have different colors. For a Borel graph, we define similarly its **Borel edge chromatic number**,  $\chi'_B(G)$  (and  $\chi'_\mu(G)$ ).

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Thus, surprisingly, the optimal value in the Borel problem is  $2d - 1$  instead of  $d + 1$  colors.

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## Theorem (Csóka-Lippner-Pikhurko, 2014)

*Let  $G$  be a Borel graph of degree  $\leq d$  and let  $\mu$  be such that  $G$  is measure-preserving. Then*

- $\chi'_\mu(G) \leq d + O(\sqrt{d})$
- *If  $G$  is bipartite, then  $\chi'_\mu(G) \leq d + 1$*

A **matching** in a graph  $G = (V, E)$  is a set  $M$  of edges that have no common vertex. For a matching  $M$  denote by  $V_M$  the set of matched vertices and call  $M$  a **perfect matching** if  $V_M = V$ . If a measure  $\mu$  on  $V$  is present and  $V_M$  has full measure, we say that  $M$  is a perfect matching  $\mu$ -a.e.

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## Theorem (König)

*Every  $d$ -regular bipartite graph has a perfect matching, for any  $d \geq 2$ .*

# A. Miller's Problem

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## Problem (A. Miller)

*Let  $G = (V, E)$  be a Borel  $d$ -regular, Borel bipartite graph. Is it true that  $G$  has a Borel perfect matching?*

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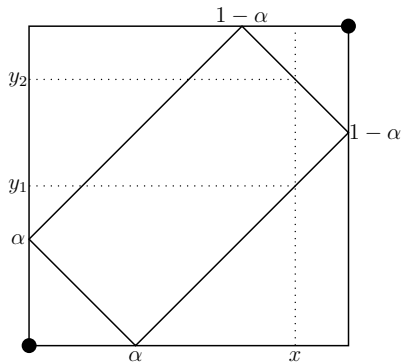
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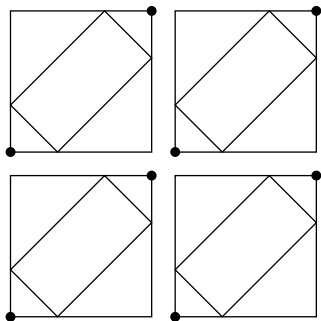
Here is his example:

Fix an irrational  $0 < \alpha < 1$  and consider the set consisting of the following rectangle inscribed in the unit square, together with the indicated two corner points.



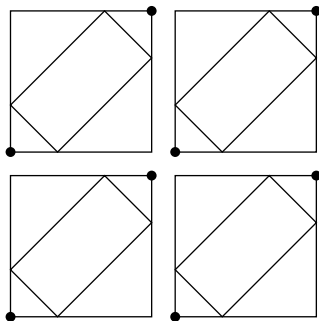
# The even $d$ case

In a paper in the early 2000s it was “shown” that putting together 4 copies of the preceding graph would produce examples for  $d = 4$  (and similarly for any even  $d$ ).



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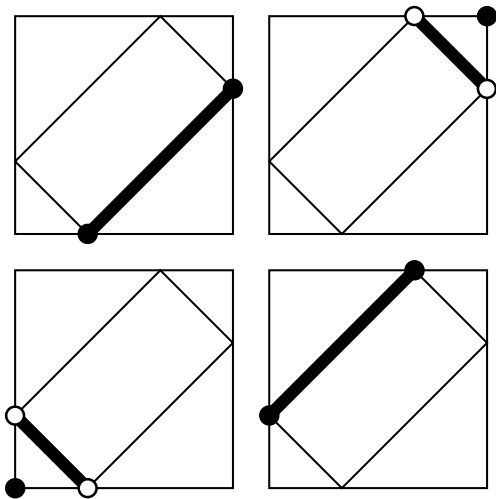
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But around 2009 Lyons showed that this did not work as this graph had a Borel matching.

# The even $d$ case

Here is a simple perfect matching found later by Conley-K:



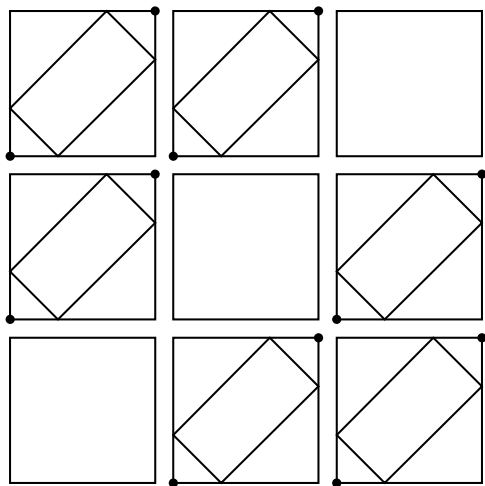
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Luckily Conley-K found a way to salvage this approach by using a “Sudoku” version:



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# The general $d$ case

These ideas do not work for odd  $d$ , so Conley-K (2009) suggested a different approach based also on ergodic theory. Let  $\mathbb{Z}_d$  be the cyclic group of order  $d$ , let  $A = B = \mathbb{Z}_d$  and consider the free part of the shift action of  $A * B$  on  $[0, 1]^{A*B}$ . This gives a  $d$ -regular, Borel acyclic, Borel bipartite graph  $G_d$  with the one side of the graph consisting of the  $A$ -orbits and the other side consisting of the  $B$ -orbits. Two such orbits are connected by an edge iff they intersect.

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It was hoped that these ergodic theory arguments would carry over to every  $d$  but this hope was dashed by a later result of Lyons-Nazarov that showed that for  $d = 3$  there is indeed a Borel perfect matching a.e.

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*Let  $(\Gamma, S)$  be a non-amenable marked group with bipartite Cayley graph. Then  $G_\infty(\Gamma, S)$  admits a Borel perfect matching a.e. (with respect to the usual product measure).*

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So ergodic theory cannot work to show that the graphs  $G_d$  admit no perfect matching for all  $d$ . However Marks recently used completely different methods, employing infinite games and Martin's Borel Determinacy Theorem, to finally show the following:

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## Remark

*Borel determinacy needs quite a bit of set theoretic power as it uses (necessarily) the existence of sets of size at least the  $\aleph_1$  iteration of the power set operation. Therefore, strangely, the only known proof of the preceding theorem needs to make use of these very large sets. The same comment applies to Marks' calculation of the Borel chromatic number of  $G_\infty(\mathbb{F}_n)$ .*

# Applications to paradoxical decompositions

There is a close connection between matchings and paradoxical decompositions. Thus some of the results on matchings in descriptive graph combinatorics have applications in the theory of paradoxical decompositions. I will mention below some very recent work in this area.

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*Let  $A, B$  be Lebesgue measurable subsets of  $\mathbb{R}^n$  with  $n \geq 3$ . Suppose they are bounded and have nonempty interior. They are equidecomposable by rigid motions using Lebesgue measurable pieces iff they have the same measure.*

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*Let  $A, B$  be Lebesgue measurable subsets of  $\mathbb{R}^n$  with  $n \geq 1$ . Suppose they are bounded and have the same positive Lebesgue measure and their boundaries have box dimension less than  $n$ . Then they are equidecomposable by translations using Lebesgue measurable pieces.*



In the early 1990's Dougherty and Foreman solved Marczewski's Problem (from the 1930's) by showing that the Banach-Tarski Paradox can be performed using pieces with the Property of Baire. Their proof was based on the following result:

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Another proof of this result has been recently found by K-Marks (2015) using ideas concerning matchings in descriptive graph combinatorics.

# Applications to paradoxical decompositions

The classical Hall Theorem about matchings states the following:

## Theorem (Hall)

*Let  $G$  be a locally finite bipartite graph such that for any finite set of vertices  $F$  (contained in one piece of the bipartition), we have  $|N_G(F)| \geq |F|$ . Then  $G$  admits a perfect matching.*

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## Proposition

- *(K-Marks, 2015) For each  $n \geq 1$ , there is a bounded degree Borel bipartite graph  $G$  on a standard probability space  $(X, \mu)$  that satisfies  $|N_G(F)| \geq n|F|$ , for any finite set of vertices  $F$ , but  $G$  has no Borel perfect matching  $\mu$ -a.e.*

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- *There is a bounded degree Borel bipartite graph  $G$  on a Polish space  $X$  that satisfies  $|N_G(F)| \geq |F|$ , for any finite set of vertices  $F$ , but  $G$  has no Borel perfect matching on a comeager set.*

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## Theorem (Marks-Unger, 2015)

*Let  $G$  be a locally finite bipartite Borel graph such that for some  $\epsilon > 0$  and any finite set of vertices  $F$  (contained in one piece of the bipartition), we have  $|N_G(F)| \geq (1 + \epsilon)|F|$ . Then  $G$  admits a perfect matching on a comeager set.*

Mark and Unger then used this result to prove an ultimate form of the Dougherty-Foreman Theorem (by very different methods) and the solution of the Marczewski Problem:

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## Theorem (Marks-Unger, 2015)

*Suppose a group  $\Gamma$  acts by Borel automorphisms on a Polish space. If the action has a paradoxical decomposition, then it has one using sets with the property of Baire.*