The Dual Nature of Proof

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Outline

Proofs $\stackrel{?}{\approx}$ derivations

Proofs $\stackrel{?}{\rightsquigarrow}$ derivations

Arguments

Conclusions

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Proofs $\overset{?}{\rightsquigarrow}$ derivations

Arguments

Conclusions

Left-wing vs. right-wing

What is a proof? The question has two answers. The right wing ('right-or-wrong', 'rule-of-law') definition is that a proof is a logically correct argument that establishes the truth of a given statement. The left wing answer (fuzzy, democratic, and human centered) is that a proof is an argument that convinces a typical mathematician of the truth of a given statement

Keith Devlin, 2003, When is a proof? *MAA Online*.

Discovery vs. justification

That never any knowledge was delivered in the same order it was invented, no not in the mathematic, though it should seem otherwise in regard that the propositions placed last do use the propositions placed first for their proof and demonstration

Francis Bacon, 1603, quoted in Robert K. Merton & Elinor Barber, 2004, *The Travels and Adventures of Serendipity*, Princeton University Press

. . . the well-known difference between the thinker's way of finding this theorem and his way of presenting it before a public may illustrate the difference in question. I shall introduce the terms context of discovery and context of justification to mark this distinction. Then we have to say that epistemology is only occupied in constructing the context of justification.

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Back vs. front

So [mathematics'] "front" and "back" will be particular kinds or aspects of mathematical activity, the public and private, or the part offered to "outsiders" (down front) versus the part normally restricted to "insiders" (backstage).

In this sense of the term, the "front" of mathematics is mathematics in "finished" form, as it is presented to the public in classrooms, textbooks, and journals. The "back" would be mathematics as it appears among working mathematicians, in informal settings, told to one another in an office behind closed doors.

Reuben Hersh, 1991, Mathematics has a front and a back, *Synthese*, **88**

Assent-obtaining vs. pattern-exhibiting

If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that we can, in the last analysis, do nothing but *point*; . . . On the other hand it is not disputed that mathematics is full of proofs, of undeniable interest and importance, whose purpose is not in the least to secure conviction. Our interest in these proofs depends on their formal and aesthetic properties. Our object is *both* to exhibit the pattern and to obtain assent. We cannot exhibit the pattern completely, since it is far too elaborate; and we cannot be content with mere assent from a hearer blind to its beauty.

G. H. Hardy, 1928, Mathematical proof, *Mind*, **38**

Acceptable vs. formal

1. Formal proof: proof as a theoretical concept in formal logic (or metalogic), which may be thought of as the ideal which actual mathematical practice only approximates.
2. Acceptable proof: proof as a normative concept that defines what is acceptable to qualified mathematicians.

Gila Hanna, 1990, Some pedagogical aspects of proof, *Interchange* **21**

I take formal/acceptable to be the same distinction I make ... between Hilbert–Gödel–Tarski and Euclid–Hilbert proof. The first requires the definition of a formal syntax and rules of inference and Tarski's name is adjoined to consider semantics. The second takes place in natural language. While there are specified definitions and axioms, the rules of inference may be implicit.

John T. Baldwin, 2016, Foundations of mathematics: Reliability and clarity: The explanatory role of mathematical induction, *WoLLIC 2016* J. Väänänen, ed., Springer

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Semantic vs. syntactic

Let us fix our terminology to understand by *proof* a conceptual proof of customary mathematical discourse, having an irreducible semantic content, and distinguish it from *derivation*, which is a syntactic object of some formal system

Yehuda Rav, 1999, Why do we prove theorems? *Philosophia Mathematica* **7**

We define a syntactic proof production to occur when the prover draws inferences by manipulating symbolic formulae in a logically permissible way. We define a semantic proof production to occur when the prover uses instantiations of mathematical concepts to guide the formal inferences that he or she draws.

Keith Weber & Lara Alcock, 2004, Semantic and syntactic proof productions, *Educational Studies in Mathematics* **56**

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Rigorous vs. Hilbertian

To those brought up in a logic-based tradition there seems to be a simple and clear definition of proof. . . :

> *A proof is a sequence of formulae each of which is either an axiom or follows from earlier formulae by a rule of inference.*

Let us call a proof in this format *Hilbertian*.
But formal logic and its Hilbertian view of proof is largely a twentieth century invention. . . . Prior to the invention of formal logic, a proof was any convincing argument. Indeed, it still is. Presenting proofs in Hilbertian style has never taken off within the mathematical community. Instead, mathematicians write rigorous proofs, i.e. proofs in whose soundness the mathematical community has confidence, but which are not Hilbertian

Alan Bundy, Mateja Jamnik & Andrew Fugard, 2005, What is a proof?
Philosophical Transactions of The Royal Society A **363**

Analytic vs. axiomatic

A) The notion of axiomatic proof. Proofs are deductive derivations of propositions from primitive premisses that are true in some sense of 'true'. They start from given primitive premisses and go down to the proposition to be proved. Their aim is to give a foundation and justification of the proposition.

B) The notion of analytic proof. Proofs are non-deductive derivations of plausible hypotheses from problems, in some sense of 'plausible'. They start from a given problem and go up to plausible hypotheses. Their aim is to discover plausible hypotheses capable of giving a solution to the problem.

Carlo Cellucci, 2008, Why proof? What is a proof? In Giovanna Corsi & Rossella Lupacchini, edd., *Deduction, Computation, Experiment*, Springer

[T]he axiomatic proof, when it exists at all, is only the core of a proof. ... Analytic proofs are more basic, more interesting and correspond to the observed way of reasoning in mathematics.

Norma B. Goethe & Michèle Friend, 2010, Confronting ideals of proof with the ways of proving of the research mathematician, *Studia Logica*, **96**

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Events vs. objects

Mathematicians talk of 'proofs' as real things. But all we can ever actually find in the real world of actual experience are *proof events*, or "provings", each of which is a social interaction occurring at a particular time and place, involving particular people, who have particular skills as members of an appropriate mathematical social community.

. . .

Mathematicians habitually and professionally reify, and it seems that what they call proofs are idealized Platonic "mathematical objects," like numbers, that cannot be found anywhere on this earth, but are nevertheless real. So let us agree to go along with this confusion (I almost wrote "joke"), and call any object or process a "proof" if it effectively mediates a proof event, not forgetting that an appropriate context is also needed.

Joseph Goguen, 1999, An introduction to algebraic semiotics, with application to user interface design. In C. L. Nehaniv, ed., *Computation for Metaphors, Analogy, and Agents*, Springer

Subjective vs. objective

Bolzano distinguishes two notions of proof, that is, two notions of consequence: one objective, what he calls *grounding* or relation from ground to consequence, one subjective, the relation from *epistemic reason* to consequence. In other terms, Bolzano makes a distinction between the objective grounding of a truth, and the subjective means that enable us to know it.

...

Bolzano's programme aims at substituting for "ordinary" proofs (*Gewissmachungen*) proofs that solely imply the conviction in the truth of the propositions they prove, by means of grounding proofs (*Begründungen*) that proceed in indicating, at each stage, the propositions to which the conclusion owes its truth immediately.

Jacques Dubucs & Sandra Lapointe, 2006,
On Bolzano's alleged explicativism, *Synthese*, **150**

Doxastic vs. propositional

A belief that *h* is *doxastically justified for S* when and only when S is acting in an epistemically responsible manner in believing that *h*.

We can say that a proposition, *h*, is *propositionally justified for S* just in case there is an *epistemically adequate basis* for *h* that is available to S regardless of whether S believes that *h*, or whether S is aware that there is such a basis, or whether if S believes that *h*, then S believes *h* on that basis.

Peter Klein, 2007, Human knowledge and the infinite progress of reasoning. *Philosophical Studies* **134**

Proofs vs. Derivations

Proofs	Derivations
‘left-wing’	‘right-wing’
discovery	justification
back	front
assent-obtaining	pattern-exhibiting
acceptable	formal
Hilbert–Gödel–Tarski	Euclid–Hilbert
semantic	syntactic
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Outline

Recipe/sketch

In practice, a proof is a sketch, in sufficient detail to make possible a routine translation of this sketch into a formal proof.

Saunders Mac Lane, 1986, *Mathematics: Form and Function*. Springer

It follows from the Derivation Recipe model that P is not as it stands (before any translation) a proof of C, but is rather an argument to convince the reader that:

C' there is a suitable formal system S such that $\vdash_S \gamma$, where γ is the formula in S corresponding to C

Compare this with the more straightforward view that P is as it stands, before any translation into a formal language, an adequate proof of C. Both of these views require P in its native, untranslated state to prove a mathematical result—they just differ over whether that result is C or C'.

Brendan Larvor, 2016, Why the naïve derivation recipe model cannot explain how mathematicians' proofs secure mathematical knowledge, *Philosophia Mathematica* **24**

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Indication

This is my derivation-indicator view of ordinary mathematical proof. On that view, a standard mathematical proof *indicates* any of a family of derivations without those derivations

(1) being what standard proofs abbreviate,

(2) being, in some more extended sense, the 'logical forms' of such proofs, or

(3) being items that such proofs are 'reducible to'.

Instead, ordinary mathematical proof, by (among other things) unsystematic combinations of genuine derivation sketches, allusions to such sketches elsewhere in the literature, and meta-derivational considerations, convinces mathematicians that a derivation of such and such a sort exists.

Jody Azzouni, 2005, How to nominalize formalism, *Philosophia Mathematica*, **13**

Indication

- (Rigour) To give an account of how informal proofs are (or can be said to be) rigorous through their connection to formal proofs.
- (Correctness) To distinguish correct informal proofs from incorrect ones, i.e., the connection should only link the informal proofs that are correct to the justifying formal proofs.
- (Agreement) To explain how, in practice, mathematicians manage to converge and agree on the correctness of informal proofs consistently. (Additionally, to give an account of informal proofs that were conceived of long before we had a sufficiently strong account of formal proofs to support them.)
- (Content) To show how the content of an informal proof determines the structure of the formal proof(s) it maps to.
- (Techniques) To provide an explanation of apparently inherently informal techniques.

Fenner Tanswell, 2015, A problem with the dependence of informal proofs on formal proofs, *Philosophia Mathematica*, **23**

Explication

1. The explicatum [the thing which explicates] is to be *similar to the explicandum* [the thing requiring explication] in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however, close similarity is not required, and considerable differences are permitted.
2. The characterization of the explicatum, that is, the rules of its use (for instance, in the form of a definition), is to be given in an *exact* form, so as to introduce the explicatum into a well-connected system of scientific concepts.
3. The explicatum is to be a *fruitful* concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a nonlogical concept, logical theorems in the case of a logical concept).
4. The explicatum should be as *simple* as possible; this means as simple as the more important requirements (1), (2), (3) permit.

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Rudolf Carnap, 1950, *Logical Foundations of Probability*. Routledge & Kegan Paul

Explication

the concept *formal proof in classical first-order logic (FOL)* is an explicatum, in Carnap's sense, of the vague and ambiguous concept *informal (mathematical) proof*. Mathematical and formal proofs are, however, not two different kinds of objects. While *mathematical proof* is a vague concept in need of an explication, *formal proof* is an exact one, and the set of formal proofs is a (proper) subset of the set of mathematical ones

Jörgen Sjögren, 2010, A note on the relation between formal and informal proof, *Acta Analytica*, **25**

why not think of "formally provable(-in-T)" (for some instantiation of "T") as a Carnapian explication of "informally provable"? The answer is simple: because it is not. ... There is no reason to believe that if one could explicate informal provability at all, then this could not be done while preserving more of its essential features than any explication in terms of "formally provable(-in-T)" would ever achieve

Hannes Leitgeb, 2009, Why do we prove theorems? In Otávio Bueno and Øystein Linnebo, edd., *New Waves in Philosophy of Mathematics*, Palgrave Macmillan

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Grounding

a mathematical inference (P, C, O) is valid within a given mathematical practice $\langle L, M, Q, R, S \rangle$ if and only if the operation O provides a ground for the conclusion C given grounds for the premises P, where grounds for C and P are specified by the set of metamathematical views M. The validity of mathematical inference is thus defined by components R [the set of accepted reasonings] and M together

Yacin Hamami, 2014, Mathematical rigor, proof gap and the validity of mathematical inference, *Philosophia Scientiæ*, **18**

Evidence

A proof in a fully formal system of logic that a claim follows from some axioms is not a proof in mathematics. It is evidence that can be used in a mathematical proof.

Richard L. Epstein, 2013, Mathematics as the art of abstraction.
In Andrew Aberdein & Ian J. Dove, edd., *The Argument of Mathematics*. Springer

Outline

Epstein's Picture of Mathematical Proof

A Mathematical Proof

Assumptions about how to reason and communicate.

argument ↓

A Mathematical Inference

Premises

necessity ↓

Conclusion

The mathematical inference is valid.

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The Parallel Structure of Mathematical Proof

Argumentational Structure:

Mathematical Proof, P_n

Endoxa: Data accepted by mathematical community

$\Big\downarrow$ *argument*

Claim: I_n is sound

Inferential Structure:

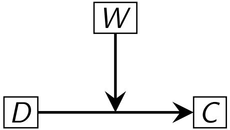
Mathematical Inference, I_n

Premisses: Axioms or statements formally derived from axioms

$\Big\downarrow$ *derivation*

Conclusion: An additional formally expressed statement

What is a Toulmin Layout?



(a) Basic Layout

(b) Full Layout

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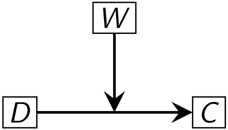

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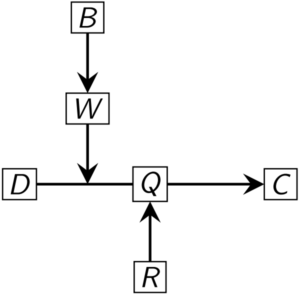

(b) Full Layout

What is a Toulmin Layout?

Given that [D] Harry was born in Bermuda, we can [Q] presumably claim that [C] he is British, since [W] anyone born in Bermuda will generally be British (on account of [B] various statutes...), unless [R] his parents were aliens, say.

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Appeal to Expert Opinion

Argumentation Scheme

Major Premise Source E is an expert in subject domain S containing proposition A.

Minor Premise E asserts that proposition A (in domain S) is true (false).

Conclusion A may plausibly be taken to be true (false).

Critical Questions

1. Expertise Question: How credible is E as an expert source?
2. Field Question: Is E an expert in the field that A is in?
3. Opinion Question: What did E assert that implies A?
4. Trustworthiness Question: Is E personally reliable as a source?
5. Consistency Question: Is A consistent with what other experts assert?

Douglas Walton, 1997, *Appeal to Expert Opinion*

Argument from Analogy

Argumentation Scheme

Similarity Premise	Generally, case C_1 is similar to case C_2.
Base Premise	A is true (false) in case C_1.
Conclusion	A is true (false) in case C_2.

Critical Questions

1. Are there differences between C_1 and C_2 that would tend to undermine the force of the similarity cited?
2. Is A true (false) in C_1?
3. Is there some other case C_3 that is also similar to C_1, but in which A is false (true)?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, Cambridge.

Three Sorts of Scheme

- **A-schemes** correspond directly to a derivation rule of the inferential structure.
- **B-schemes** are less directly tied to the inferential structure. Their instantiations correspond to substructures of derivations rather than individual derivations. B-schemes may be thought of as exclusively mathematical arguments: high-level algorithms or macros that may in principle be formalized as multiple inferential steps.
- **C-schemes** are even looser in their relationship to the inferential structure, since the link between their grounds and claim need not be deductive.

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Four Views of Mathematical Proof

0. Only A-schemes are admissible. There is no such thing as ‘informal mathematical reasoning’: only formalized reasoning can count as mathematical. All that the argumentational structure can do is ‘point’ at the inferential structure.
1. Only A- and B-schemes are admissible. Informal mathematical reasoning is possible, but the argumentational structure must employ exclusively mathematical steps, albeit ones characterized informally.
2. All three types of scheme are admissible. Informal mathematical reasoning is possible, and the argumentational structure employs both exclusively mathematical steps, and steps of more general application.
3. Only topic-neutral A- or C-schemes are admissible. Informal mathematical reasoning is possible, and must be understandable purely in terms of steps of general application. No argumentational structure need contain any exclusively mathematical steps; that is, all such steps must be reducible to instances of general steps.

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Four Views of Mathematical Proof

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1. Only A- and B-schemes are admissible. Informal mathematical reasoning is possible, but the argumentational structure must employ exclusively mathematical steps, albeit ones characterized informally.
2. All three types of scheme are admissible. Informal mathematical reasoning is possible, and the argumentational structure employs both exclusively mathematical steps, and steps of more general application.
3. Only topic-neutral A- or C-schemes are admissible. Informal mathematical reasoning is possible, and must be understandable purely in terms of steps of general application. No argumentational structure need contain any exclusively mathematical steps; that is, all such steps must be reducible to instances of general steps.

Outline

Proofs $\stackrel{?}{\approx}$ derivations

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Arguments

Conclusions

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- Mathematical reasoning is comprised of arguments.
- Some arguments are proofs; some are derivations.
- The practice of proof is comprised of two parallel structures: argumentational and inferential.
- Views about proof in the foundations of mathematics differ in terms of the admissibility of argumentation schemes as steps in the inferential structure.
- Views about proof in mathematical practice differ in terms of the admissibility of argumentation schemes as steps in the argumentational structure.
- Rigour is a relationship between the two structures. A proof is rigorous if each of the steps of its argumentational structure is related to the inferential structure in a way the mathematical audience accepts as contextually appropriate.

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