Non-deterministic Matrices in Action

Arnon Avron and Yoni Zohar

Tel Aviv Univesity

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Matrices

Logical Matrices

A matrix **M** consists of:

- a set $\mathcal{V}$ of *truth values*
- a subset $\mathcal{D}$ of *designated* truth values
- for each connective $\diamond$ of arity n, a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \rightarrow \mathcal{V}$

M-valuations

$$v(\diamond(\varphi_1, \ldots, \varphi_n)) = \mathcal{O}(\diamond)(v(\varphi_1), , \ldots, v(\varphi_n))$$

$$v(p) = a_1$$
$$v(q) = a_2$$
$$v(p \wedge q) = a_2$$
$$v(q \wedge p) = a_3$$
$$\vdots$$

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v:

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \Longrightarrow v(\varphi) \in \mathcal{D}$$

Non-deterministic Matrices

Non-deterministic Logical Matrices [Avron et al.'05]

A Nmatrix **M** consists of:

- a set $\mathcal{V}$ of *truth values*
- a subset $\mathcal{D}$ of *designated* truth values
- for each connective $\diamond$ of arity n, a "truth table" $\mathcal{O}(\diamond) : \mathcal{V}^n \to P^+(\mathcal{V})$

M-valuations

$$v(\diamond(\varphi_1, \dots, \varphi_n)) \in \mathcal{O}(\diamond)(v(\varphi_1), , \dots, v(\varphi_n))$$

$v(p) = a_1$
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$v(q \wedge p) = a_3$
$\vdots$

When Does $\mathcal{T} \vdash_{\mathbf{M}} \varphi$?

For every **M**-valuation v:

$$v(\psi) \in \mathcal{D} \text{ for every } \psi \in \mathcal{T} \implies v(\varphi) \in \mathcal{D}$$

Primal Infon Logic [Gurevich,Neeman'09]

- An extremely efficient logic
- The main logical engine in DKAL (Microsoft)

- $\mathcal{V} = \{0, 1\}$
- $\mathcal{D} = \{1\}$

$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0, 1\}$

$\mathcal{O}(\supset)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{1\}$	$\{0, 1\}$

- $v(p) = 0,\ v(q) = 0,\ v(p \vee q) = 0$ is an **M**-valuation
- $v(p) = 0,\ v(q) = 0,\ v(p \vee q) = 1$ is an **M**-valuation
- $p \vee p \not\vdash_{\mathbf{M}} p$
- No finite matrix

Rexpansions: Refined Expansions

$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$, $\mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$

Expansion

$\mathbf{M}_2$ is an expansion of $\mathbf{M}_1$ if $\exists\ F$ s.t.:

- $\mathcal{V}_2 = \biguplus_{x \in \mathcal{V}_1} F(x)$
- $\mathcal{D}_2 = \biguplus_{x \in \mathcal{D}_1} F(x)$
- $\mathcal{O}_2(\diamond)(\overline{y}) = \biguplus_{z \in \mathcal{O}_1(\overline{x})} F(z)$,

Refinement

$\mathbf{M}_2$ is a refinement of $\mathbf{M}_1$ if:

- $\mathcal{V}_2 \subseteq \mathcal{V}_1$
- $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$
- $\mathcal{O}_2(\diamond)(\overline{x}) \subseteq \mathcal{O}_1(\diamond)(\overline{x})$

Proposition [Avron et al. '07,'11]

- If $\mathbf{M}_2$ is an expansion of $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.
- If $\mathbf{M}_2$ is a refinement of $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} \subseteq \vdash_{\mathbf{M}_2}$.

Rexpansions: Refined Expansions

$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$, $\mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$

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- $\mathcal{O}_2(\diamond)(\overline{x}) \subseteq \mathcal{O}_1(\diamond)(\overline{x})$

$\mathbf{M}_2$ is a rexpansion of $\mathbf{M}_1$ if it is a refinement of some expansion of $\mathbf{M}_1$

Rexpansions: Refined Expansions

$\mathbf{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$, $\mathbf{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$

Expansion

$\mathbf{M}_2$ is an expansion of $\mathbf{M}_1$ if $\exists$ F s.t.:

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- $\mathcal{D}_2 = \biguplus_{x \in \mathcal{D}_1} F(x)$
- $\mathcal{O}_2(\diamond)(\overline{y}) = \biguplus_{z \in \mathcal{O}_1(\overline{x})} F(z)$,

Refinement

$\mathbf{M}_2$ is a refinement of $\mathbf{M}_1$ if:

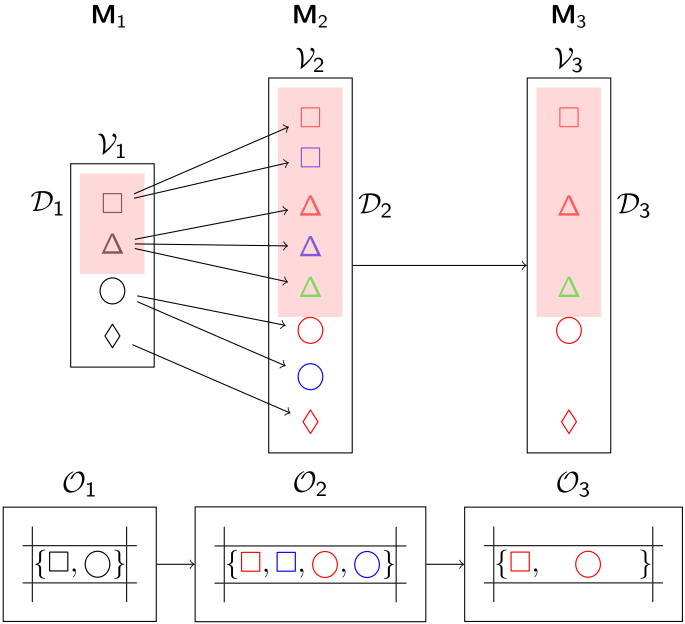
- $\mathcal{V}_2 \subseteq \mathcal{V}_1$
- $\mathcal{D}_2 = \mathcal{V}_2 \cap \mathcal{D}_1$
- $\mathcal{O}_2(\diamond)(\overline{x}) \subseteq \mathcal{O}_1(\diamond)(\overline{x})$

$\mathbf{M}_2$ is a rexpansion of $\mathbf{M}_1$ if it is a refinement of some expansion of $\mathbf{M}_1$

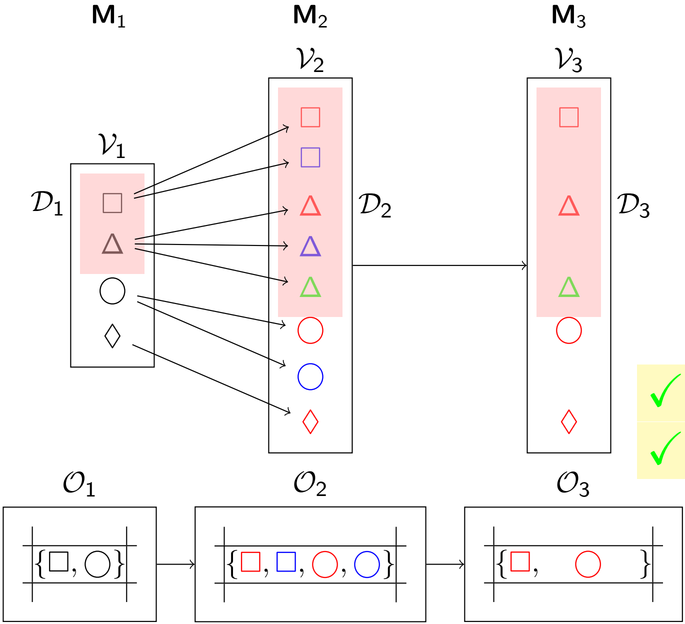
$\mathbf{M}_2$ is a preserving rexpansion of $\mathbf{M}_1$ if $F(x) \cap \mathcal{V}_2 \neq \emptyset$ for every $x \in \mathcal{V}_1$

$\mathbf{M}_2$ is a preserving rexpansion of $\mathbf{M}_1$, if in addition:
$F(z) \cap \mathcal{O}_2(\diamond)(y_1, \ldots, y_n) \neq \emptyset$

(Strongly) Preserving Rexpansions



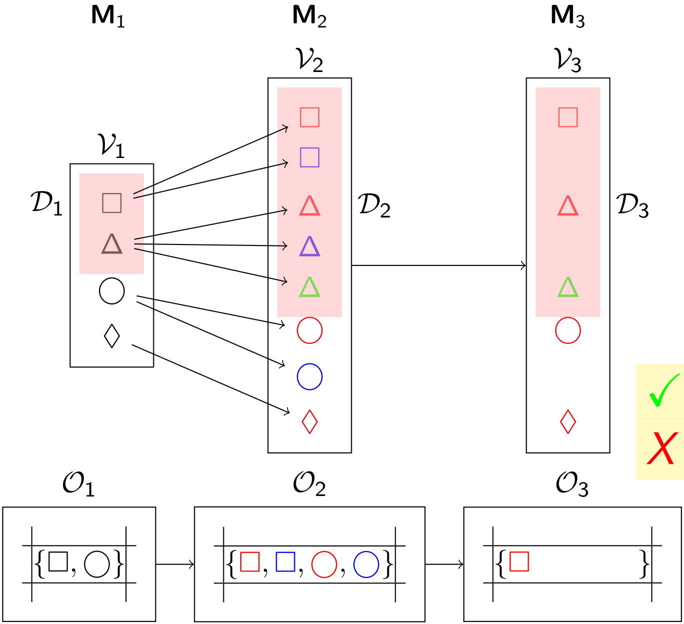
(Strongly) Preserving Rexpansions



✓ preserving

✓ strongly preserving

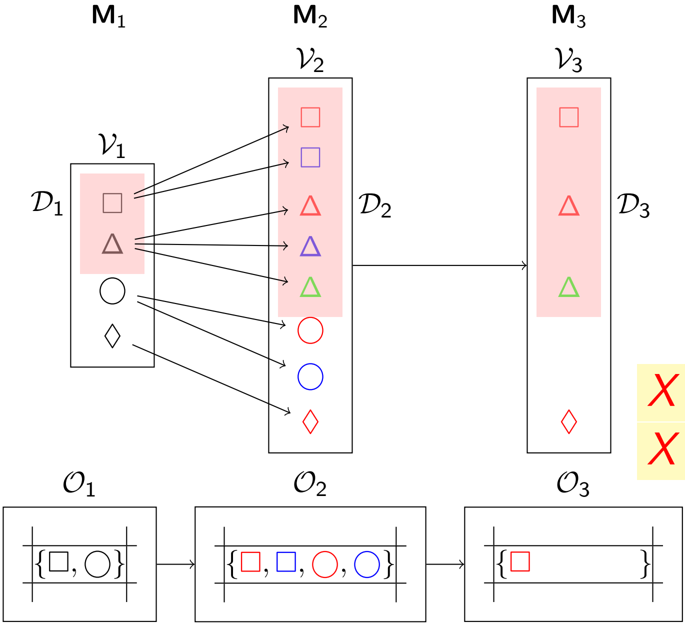
(Strongly) Preserving Rexpansions



✓ preserving

X strongly preserving

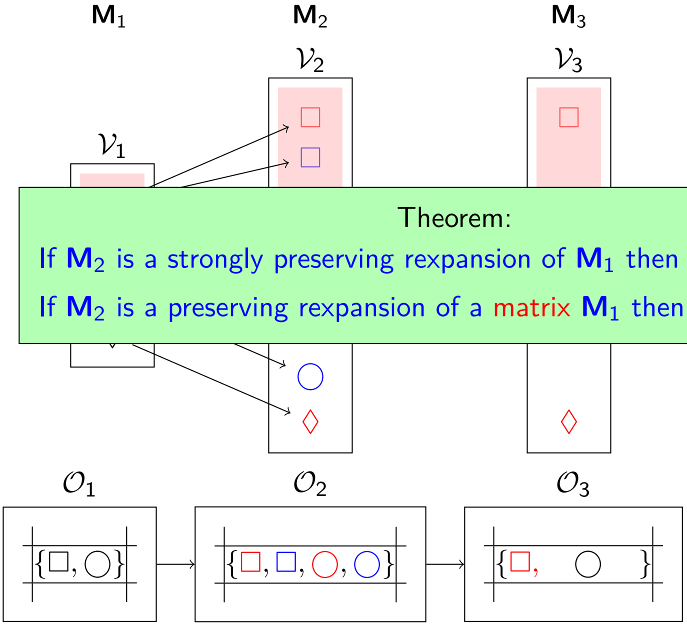
(Strongly) Preserving Rexpansions



X preserving

X strongly preserving

(Strongly) Preserving Rexpansions



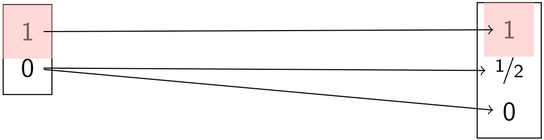
Theorem:

If $\mathbf{M}_2$ is a strongly preserving rexpansion of $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.

If $\mathbf{M}_2$ is a preserving rexpansion of a matrix $\mathbf{M}_1$ then $\vdash_{\mathbf{M}_1} = \vdash_{\mathbf{M}_2}$.

Three-valued Logics

Classical Logic | Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

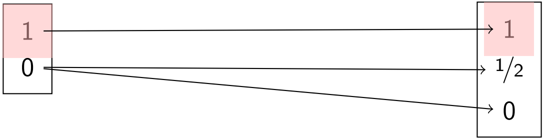
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{0, 1/2\}$	$\{0, 1/2\}$
$1/2$	$\{0, 1/2\}$	$\{0, 1/2\}$	$\{0, 1/2\}$
0	$\{0, 1/2\}$	$\{0, 1/2\}$	$\{0, 1/2\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{0, 1/2\}$	$\{0, 1/2\}$
0	$\{1\}$	$\{0, 1/2\}$	$\{0, 1/2\}$

Three-valued Logics

Classical Logic | Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

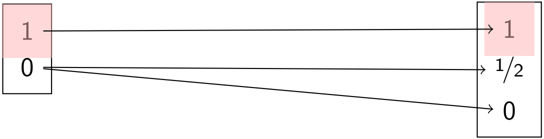
$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

Three-valued Logics

Classical Logic — Kleene Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

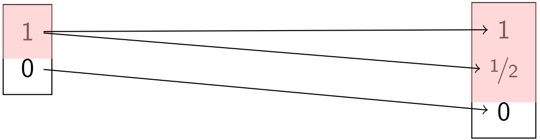
$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Therefore, Kleene logic is conservative over $\wedge\vee$-classical logic

Three-valued Logics

Classical Logic | Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1,1/2\}$	$\{1,1/2\}$	$\{0\}$
$1/2$	$\{1,1/2\}$	$\{1,1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

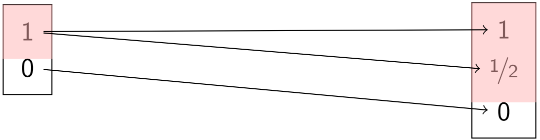
$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1,1/2\}$	$\{1,1/2\}$	$\{1,1/2\}$
$1/2$	$\{1,1/2\}$	$\{1,1/2\}$	$\{1,1/2\}$
0	$\{1,1/2\}$	$\{1,1/2\}$	$\{0\}$

Three-valued Logics

Classical Logic

Priest Logic

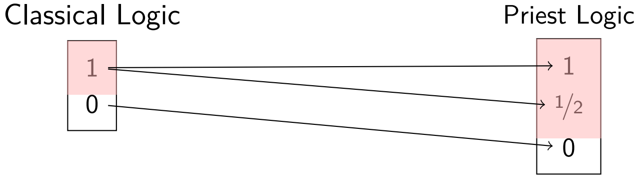


$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

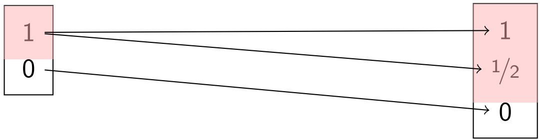
$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Three-valued Logics

Classical Logic — Priest Logic



$\mathcal{O}(\wedge)$	1	0
1	$\{1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	0
1	$\{1\}$	$\{1\}$
0	$\{1\}$	$\{0\}$

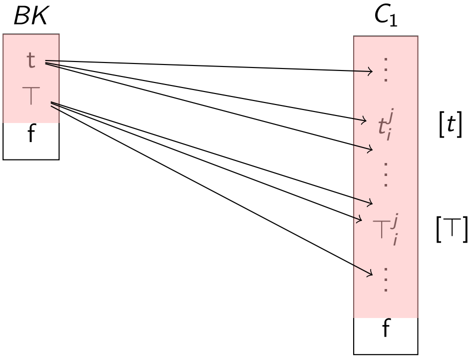
$\mathcal{O}(\wedge)$	1	$1/2$	0
1	$\{1\}$	$\{1/2\}$	$\{0\}$
$1/2$	$\{1/2\}$	$\{1/2\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\mathcal{O}(\vee)$	1	$1/2$	0
1	$\{1\}$	$\{1\}$	$\{1\}$
$1/2$	$\{1\}$	$\{1/2\}$	$\{1/2\}$
0	$\{1\}$	$\{1/2\}$	$\{0\}$

	$\mathcal{O}(\neg)$
1	$\{0\}$
$1/2$	$\{1/2\}$
0	$\{1\}$

Therefore, Priest logic is conservative over $\wedge\vee$-classical logic

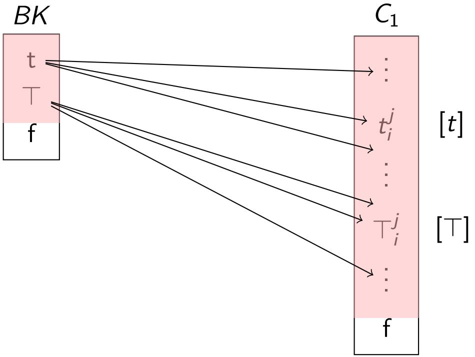
Paraconsistent Logics [Avron'07]



	$\mathcal{O}(\neg)$
t	{f}
T	{t,T}
f	{t,T}

$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ [t] \cup [\top] & x = \top_i^j \\ [t] \cup [\top] & x = f \end{cases}$$

Paraconsistent Logics [Avron'07]



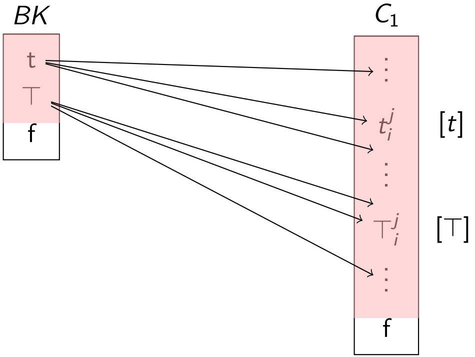
✓ preserving

✓ strongly preserving

	$\mathcal{O}(\neg)$
t	{f}
⊤	{t, ⊤}
f	{t, ⊤}

$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ \left\{\top_i^{j+1}, t_i^{j+1}\right\} & x = \top_i^j \\ [t] \cup [\top] & x = f \end{cases}$$

Paraconsistent Logics [Avron'07]



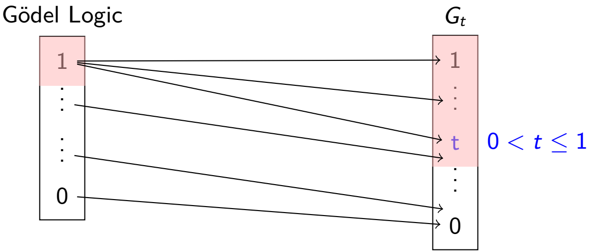
✓ preserving

✗ strongly preserving

	$\mathcal{O}(\neg)$
t	{f}
⊤	{t,⊤}
f	{t,⊤}

$$\mathcal{O}(\neg)(x) = \begin{cases} \{f\} & x = t_i^j \\ \left\{\top_i^{j+1}, t_i^{j+1}\right\} & x = \top_i^j \\ [t] & x = f \end{cases}$$

Gödel Logic



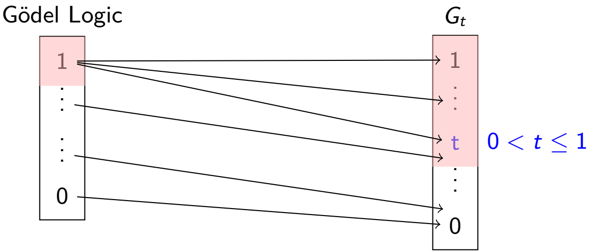
$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\} \qquad \mathcal{O}(\wedge)(x,y) = \begin{cases} [t,1] & \max(x,y) \geq t \\ \{\max(x,y)\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\} \qquad \mathcal{O}(\vee)(x,y) = \begin{cases} [t,1] & \min(x,y) \geq t \\ \{\min(x,y)\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & \textit{otherwise} \end{cases} \qquad \mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & \textit{otherwise} \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\} \qquad \mathcal{O}(\bot)(x,y) = \{0\}$$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

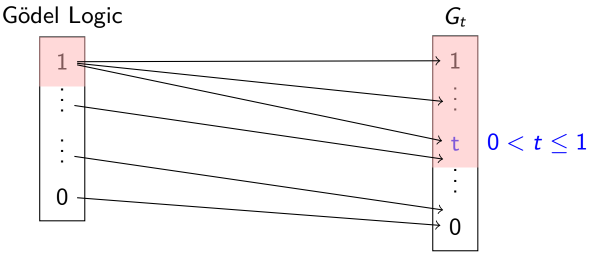
$$\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

$$\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$$

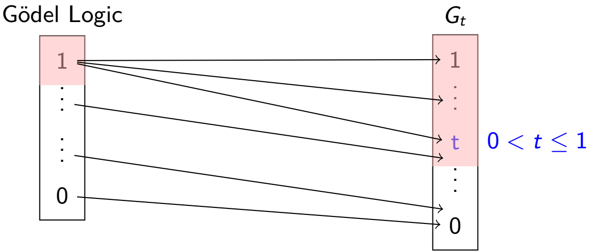
$$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

In Gödel logic, $\mathcal{D}$ may be $[t, 1]$

Gödel Logic



$$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$$

$$\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$$

$$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$$

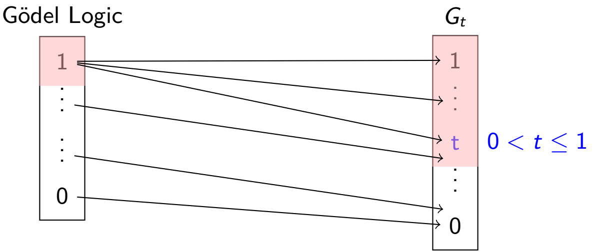
$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

$$\mathcal{O}(\bot)(x,y) = \{0\}$$

$$\mathcal{O}(\neg)(x) = 1 - x$$



$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$

$\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

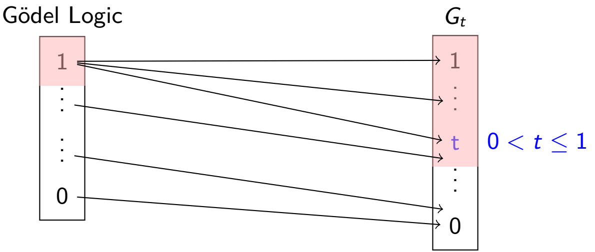
$\mathcal{O}(\perp)(x,y) = \{0\}$

$\mathcal{O}(\perp)(x,y) = \{0\}$

$\mathcal{O}(\neg)(x) = 1 - x$

All refinements are conservative!

Gödel Logic



$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases}$$

$\mathcal{O}(\bot)(x,y) = \{0\}$

$\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

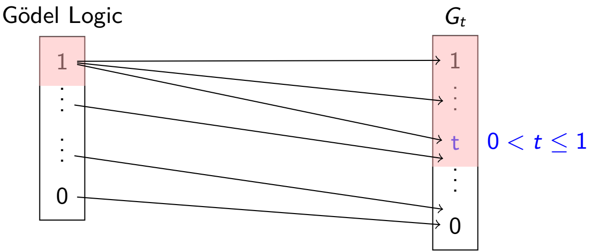
$\mathcal{O}(\bot)(x,y) = \{0\}$

$\mathcal{O}(\neg)(x) = 1 - x$

Exactly three:

$\vdash_{G_{1/2}}, \vdash_{G_{3/4}}, \vdash_{G_1}$

Gödel Logic



$\mathcal{O}(\wedge)(x,y) = \{\min(x,y)\}$ $\qquad$ $\mathcal{O}(\wedge)(x,y) = \{max(x,y)\}$

$\mathcal{O}(\vee)(x,y) = \{\max(x,y)\}$ $\qquad$ $\mathcal{O}(\vee)(x,y) = \{max(x,y)\}$

$$\mathcal{O}(\supset)(x,y) = \begin{cases} \{1\} & x \leq y \\ \{y\} & otherwise \end{cases} \qquad \mathcal{O}(\supset)(x,y) = \begin{cases} [t,1] & x \leq y \text{ or } y \geq t \\ \{y\} & otherwise \end{cases}$$

$\mathcal{O}(\bot)(x,y) = \{0\}$ $\qquad$ $\mathcal{O}(\bot)(x,y) = \{0\}$

$\mathcal{O}(\neg)(x) = 1 - x$

for $t \leq 1/2$: $\vdash_{G_t}$ is paraconsistent

Conclusion

Applications of Rexpansions:

- A sufficient criterion for conservative extensions
- Particular instances: paraconsistent (and) fuzzy logics

Future work:

- A necessary criterion for conservative extensions:
  - $\wedge\vee$-Kleene = $\wedge\vee$-Priest logic
  - None is (even) a rexpansion of the other
- Beyond Gödel logic
  - Łukasiewicz logic, product logic
  - Arbitrary t-norm based logics

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Thank you!