Eilenberg-MacLane spaces in Homotopy Type Theory

Floris van Doorn

Carnegie Mellon University

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j.w.w. Ulrik Buchholtz (TU Darmstadt) and Egbert Rijke (CMU)
There are models of type theory in various abstract frameworks for homotopy theory.

Examples:

- Quillen model categories [Awodey, Warren, 2009];
- Simplicial sets [Streicher, 2011];
- Cubical sets [Bezem, Coquand, Huber, 2014];
- ... and many more.
This leads to a new program, *Synthetic Homotopy Theory*:

Study types in type theory as spaces in homotopy theory.

This gives a more general and constructive treatment of homotopy theory which is easier to verify formally in a computer proof assistant.

▶ The main theorem in this talk has been fully formalized.

I work in *Homotopy Type Theory* (HoTT): dependent type theory with *univalence* and *higher inductive types* [Homotopy Type Theory, 2013].

As motivating example I will concentrate on *Eilenberg-MacLane spaces*. 
Homotopy Type Theory

Homotopy Type Theory combines Type Theory with Homotopy Theory.

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<th>Logic</th>
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<td>$a : A$</td>
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<td>$P : A \to \text{Type}$</td>
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<td>$\Sigma(x : A). P(x)$</td>
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<td>$a =_A b$</td>
<td>Identity Type</td>
<td>Equality</td>
<td>Path space</td>
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I will use these notions interchangably.
A type $A$ can have
- points $a, b : A$
- paths $p, q : a = b$
- paths between paths $r : p = q$
Identity Type

Different ways to think about the identity type:

- **Type theory**: The identity type is generated by reflexivity:
  \[ \text{refl}_a : a =_A a. \]

- **Logic**: Equality is the least (free) reflexive relation.

- **Homotopy theory**: The path space with one point fixed is contractible.

(This does not mean every proof of equality is reflexivity)
Path Induction

This is made precise by path induction:

- If $C : \Pi(x : A). \; a = x \to \text{Type}$,
- to prove/construct an element of $\Pi(x : A).\Pi(p : a = x). \; C(x, p)$
- it is sufficient to prove/construct an element of $C(a, \text{refl}_a)$

Example Symmetry of equality (invertibility of paths)

$$\Pi(A : \text{Type}). \Pi(a \; b : A). \; a = b \to b = a.$$  

Proof. Suppose $A$ is a type and $a : A$. We need to prove $\Pi(b : A). \; a = b \to b = a$.
We apply path induction, in which case we need to prove $a = a$, which is true by $\text{refl}_a$. 

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We can also look at the identity type in a type-oriented way:

$$(a, b) =_{A \times B} (a', b') \quad \text{is} \quad (a =_A a') \times (b =_B b')$$

$$f = g \quad \text{is} \quad \Pi x. \; f(x) = g(x) \quad \text{(function extensionality)}$$

$$A =_{\text{Type}} B \quad \text{is} \quad A \simeq B \quad \text{(univalence, Voevodsky)}$$

This is done in \textit{cubical type theory}. 
Truncated Types

Some types are *truncated*, which means there are all higher paths are trivial.

A type $A$ is *contractible* ($(-2)$-type) if it has exactly one element, if

$$\Sigma(x : A). \Pi(y : A). x = y.$$  

A type $A$ is a *proposition* ($(-1)$-type) if it has at most one element, if

$$\Pi(x \ y : A). x = y.$$  

In either of the above cases, all (higher) paths in $A$ are trivial.

$A$ is a *set* ($0$-type) if for all $x \ y : A$ the type $x = y$ is a proposition.

$A$ is an $(n + 1)$-type if for all $x \ y : A$ the type $x = y$ is an $n$-type.
Truncated Types
Given $A$, we can form the $n$-truncation $\|A\|_n$.

$\|A\|_n$ is the “best approximation” of $A$ which is $n$-truncated.

If $X$ is $n$-truncated, we get the following universal property:

$$
\begin{array}{c}
A \\
\downarrow \ |-|_n \\
\|A\|_n \\
\end{array} 
\xrightarrow{\forall} 
\begin{array}{c}
\|A\|_n \\
\xrightarrow{\exists!} 
X \\
\end{array}
$$
Higher Inductive Types

In Type Theory there are *inductive types*, in which you specify its points.

**Examples.** $\mathbb{N}$ is generated by 0 and $\text{succ}$

- $A + B$ is generated by either $a : A$ or $b : B$
- $a =_A (-)$ is generated by $\text{refl}_a : a =_A a$

In homotopy theory we can build cell complexes inductively.

In HoTT we can combine these into *higher inductive types* [Shulman, Lumsdaine, 2012].
The circle

**Example.** The circle $S^1$

\[
\text{HIT } S^1 :=
\begin{align*}
\bullet & \text{ base : } S^1 \\
\bullet & \text{ loop : base } = \text{ base }
\end{align*}
\]

Using univalence, we can prove $\text{loop } \neq \text{ refl}$.

**Recursion Principle.** To define $f : S^1 \to A$ we need to define $a : A$ and $p : a = a$. 
Example. The suspension $\Sigma A$

HIT $\Sigma A :=$

- north, south : $\Sigma A$
- merid : $A \to (\text{north} = \text{south})$

Remark. $S^1 \simeq \Sigma 2$

Definition. We can now define the $n$-spheres by $S^{n+1} := \Sigma S^n$ and $S^0 := 2$
Homotopy Groups

In algebraic topology, we look for algebraic invariants of spaces, like the homotopy groups.

Traditionally: \( \pi_n(A, a_0) = \{ f : S^n \to A \mid f \text{ preserves basepoints} \} / \sim. \)

In HoTT \( \pi_n(A, a_0) = \| S^n \to^* A \|_0 \) where we use \( \to^* \) for basepoint preserving maps.

Alternative characterization: \( \pi_n(A, a_0) = \| \Omega^n(A, a_0) \|_0 \) where \( \Omega(A, a_0) = (a_0 = a_0, \text{refl}_{a_0}) \).

These are groups for \( n \geq 1 \) (abelian for \( n \geq 2 \)).
If $X$ is $n$-truncated then $\pi_k(X) = 0$ for all $k > n$.

The converse is not true in general.

**Definition.** A type $A$ is $n$-connected if $\|A\|_n$ is contractible.

**Remark.** ($-1$)-connected: merely inhabited; 0-connected: path-connected; 1-connected: simply connected.

$X$ is $n$-connected if and only if $\pi_k(X) = 0$ for all $k \leq n$.

If $X$ is $n$-connected, then $\Sigma X$ is $(n + 1)$-connected.

Thus the $n$-sphere $\mathbb{S}^n$ is $(n - 1)$-connected.
Homotopy Groups of spheres

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Eilenberg MacLane spaces

**Question.** Can we construct spaces with simple homotopy groups?

In classical homotopy theory, the *Eilenberg MacLane space* $K(G, n)$ is the unique space such that

$$
\pi_n(K(G, n)) = \begin{cases} 
G & \text{if } n = k \\
0 & \text{if } n \neq k.
\end{cases}
$$

We have already seen one example $K(\mathbb{Z}, 1) = S^1$.

Eilenberg-MacLane spaces classify homology and cohomology.

These can be constructed in HoTT [Licata, Finster, 2014].

We write $\text{Type}^=_{n}$ for the universe of pointed $(n - 1)$-connected $n$-truncated types. We want to construct $K(G, n) : \text{Type}^=_{n}$. 
Eilenberg MacLane space \( K(G, 1) \)

For \( n = 1 \), suppose \( G \) is a group.

\[
\text{HIT } \tilde{K}(G, 1) :=
\]
\[
\begin{itemize}
  \item \( \ast : \tilde{K}(G, 1) \)
  \item \( \text{pth} : G \to (\ast = \ast) \)
  \item \( \text{pth-mul} : \Pi(g \ h : G). \ pth(gh) = \text{pth}(g) \cdot \text{pth}(h) \)
\end{itemize}
\]

\( \tilde{K}(G, 1) \) is not quite an Eilenberg-MacLane space; it has nontrivial higher structure.

\[
K(G, 1) := \| \tilde{K}(G, 1) \|_1
\]

\( K(G, 1) \) is 0-connected, 1-truncated (it lives in Type\(^=1_\ast \) and one can show that \( \pi_1(K(G, 1)) = G \).
Eilenberg MacLane space \( K(G, n) \)

Suppose \( n \geq 1 \). We want to construct \( K(G, n + 1) \) out of \( K(G, n) \). This only works if \( G \) is abelian.

**Definition.** \[ K(G, n + 1) := \|\Sigma K(G, n)\|_{n+1} \]

Now \( K(G, n + 1) \) is indeed \( n \)-connected and \((n + 1)\)-truncated (it lives in \( \text{Type}_{\ast^{=n+1}} \)). We can show that \( \Omega K(G, n + 1) = K(G, n) \) (if \( G \) is abelian). Hence

\[
\Omega^{n+1} K(G, n + 1) = \Omega^n \Omega K(G, n + 1) = \Omega^n K(G, n) = G
\]

So \( K(G, n) \) has the right homotopy groups.
Main result

**Theorem.** Any \(X : \text{Type}_{\ast}^n\) is equivalent to \(K(\pi_n(X), n)\).

Moreover, \(K(-, n)\), interpreted as a functor from \(\text{AbGrp} \to \text{Type}_{\ast}^n\) is an equivalence of categories for \(n \geq 2\).

For \(n = 1\) it is an equivalence of categories \(\text{Grp} \to \text{Type}_{\ast}^1\).

This means that not only every \(X : \text{Type}_{\ast}^n\) is an Eilenberg-MacLane space, but also any map \(f : X \to Y\) is given by the action of a unique group homomorphism on Eilenberg MacLane spaces.
Special case: uniqueness of $K(G, 1)$

As a special case we show: if $(X, x_0) : \text{Type}_*^1$ and we have a group isomorphism $e : G \cong \pi_1(X)$ then $K(G, 1) \cong X$.

HIT $\tilde{K}(G, 1) :=$

- $\ast : \tilde{K}(G, 1)$
- $\text{pth} : G \to \ast = \ast$
- $\text{pth-mul} : \Pi(g, h : G). \text{pth}(gh) = \text{pth}(g) \cdot \text{pth}(h)$

$K(G, 1) = \|\tilde{K}(G, 1)\|_1$

Recursion Principle. To define $f : K(G, 1) \to A$ for a 1-type $A$ we need $a : A$ and $p : G \to a = a$ such that $p(gh) = p(g) \cdot p(h)$.

We define a map $f : K(G, 1) \to X$ by sending $\ast$ to $x_0$, $\text{pth}(g)$ to $e(g)$, viewed as element of $\Omega X$, and $e(gh) = e(g)e(h)$ because $e$ is a group homomorphism. Is $f$ an equivalence?
Special case: uniqueness of $K(G,1)$

$f$ induces an isomorphism on $\pi_k(K(G,1)) \to \pi_k(X)$ for all $k$ (trivially for $k \neq 1$).
Such an $f$ is called a weak equivalence.
For $k = 1$, we use that the following triangle commutes:

\[
\begin{array}{ccc}
\pi_1(K(G,1)) & \xrightarrow{\pi_1(f)} & \pi_1(X) \\
\downarrow & & \downarrow \\
G & \xrightarrow{e} & e
\end{array}
\]

**Theorem.** (Whitehead) If $g : A \to B$ is a weak equivalence, $A$ and $B$ are $n$-types for some $n$, then $f$ is an equivalence.

Hence $f : K(G,1) \to X$ is an equivalence.
This result is formally proven in the proof assistant *Lean*.

Lean is a new open source proof assistant with support for HoTT, similar to Coq and Agda.

Lean implements dependent type theory with a hierarchy of (non-cumulative) universes and inductive types (à la Dybjer, with constructors and recursors).

The kernel is smaller and simpler than those of Coq and Agda.

Lean has two modes: a standard mode for classical and constructive reasoning and a HoTT mode for Homotopy Type Theory.
The Lean HoTT library contains an extensive collection of basic concepts, and the following results were formalized:

- The Freudenthal Suspension Theorem
- The Hopf fibration
- The long exact sequence of homotopy groups
- The Seifert-van Kampen theorem
- The adjunction between the smash product and pointed maps
- Eilenberg MacLane spaces

Currently I’m working in a group project to formalize spectral sequences in Lean.
definition KG1_map {G : Group} {X : Type*} (e : G → Ω X) (r : Πg h, e (g * h) = e g · e h) [is_conn 0 X] [is_trunc 1 X] : K G 1 → X :=

begin
  intro x, induction x using EM.elim,
  { exact Point X },
  { exact e g },
  { exact r g h }
end

definition Grp_equivalence : Grp ≃ c cType*[1] :=
equivalence.mk EM1_cfunctor is_equivalence_EM1_cfunctor

definition AbGrp_equivalence (n : ℕ) : AbGrp ≃ c cType*[n+2] :=
equivalence.mk (EM_cfunctor (n+2)) (is_equivalence_EM_cfunctor n)
Advantages of Synthetic homotopy theory:

- More general
  - There are multiple models of HoTT;
- The homotopy theoretic notions are primitives in type theory
  - We don’t have to talk about topology, continuity, . . . .
- Novel ways of reasoning
  - Path induction, homotopy invariance;
- Constructive (but not anti-classical)
  - Has computational interpretation;
- Possible to verify formally in practice
  - Proof fully formalized in Lean.
Thank you

The Lean HoTT library is available at:
https://github.com/leanprover/lean2/blob/master/hott/hott.md