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Computable Structure Theory, Part I

Complexity of Structures and Relations on Structures

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Complexity of Atomic and Elementary Diagrams

Consider *countable* structures A for *computable* languages L.

- *Atomic diagram of* A, $D(A)$, is the set of all quantifier-free sentences of L_A true in A_A.

- *Turing degree* of A is the Turing degree of $D(A)$.
  A is *computable* (*recursive*) if its Turing degree is $\mathbf{0}$.

- $D(A)$ may be of much lower Turing degree than $Th(A)$.
  $\mathcal{N}$, the standard model of arithmetic, is computable.
  True Arithmetic, $TA = Th(\mathcal{N})$, is of Turing degree $\mathbf{0}^{(\omega)}$.

  $\emptyset'$ is the halting set and $\mathbf{0}'$ is its Turing degree.

- (Tennenbaum) If A is a nonstandard model of *Peano Arithmetic* (PA), then A is not computable.

- $\leq_T$ Turing reducibility
  A set X and its Turing degree $\mathbf{x}$ are called *low* if $\mathbf{x}' = \mathbf{0}'$.

- (Knight) If A is a nonstandard model of PA, then there exists $B \cong A$ such that $D(B) <_T D(A)$.

- (Harrington, Knight) There is a nonstandard model M of PA such that $D(M)$ is *low* and $Th(M) \equiv_T \emptyset^{(\omega)}$.

- (Downey and Jockusch) Every Boolean algebra of *low* Turing degree has a computable copy.

- Abelian p-groups (p is a prime number)
  $(g \in G - \{0\}) \Rightarrow (\exists m \geq 1)[o(g) = p^m]$

  $\mathbb{Z}(p^n)$ the cyclic group of order p^n
  $\mathbb{Z}(p^\infty)$ the quasicyclic (Prüfer) p-group
  the set of rationals in $[0, 1)$ of the form $\frac{i}{p^k}$ with addition modulo 1

- Descending chain of subgroups defined recursively
  $G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_\alpha \supsetneq G_{\alpha+1} \supsetneq \cdots \supsetneq G_{\lambda(G)}$
  $G_{\alpha+1} = pG_\alpha$
  α limit: $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$

- *length* of $G = \lambda(G) = \mu\lambda[G_\lambda = G_{\lambda+1}]$
  $G_\alpha - G_{\alpha+1} =$ elements of *height* α

- G a countable abelian p-group
  $\Rightarrow \lambda(G)$ is a countable ordinal

- G a computable abelian p-group
  $\Rightarrow \lambda(G) \leq \omega_1^{CK}$

- G *reduced* $\Leftrightarrow G_{\lambda(G)} = \{0\}$

- G a computable reduced abelian p-group
  $\Rightarrow \lambda(G) < \omega_1^{CK}$

- $D(G) = G_{\lambda(G)} \neq \{0\}$ (unique) *divisible* part

- $G = D(G) \oplus G_{reduced}$

- Suppose that G is a computable abelian p-group isomorphic to $\oplus_\alpha \mathbb{Z}(p^\infty) \oplus \oplus_{i<\omega} \mathbb{Z}(p^{n_i})$. Then the *character* of G,

$$\chi(G) = \{(k,n) : card(\{i : n_i = k\}) \geq n\},$$

is a $\mathbf{\Sigma}^0_2$ set.

- $K \subseteq (\omega - \{0\}) \times (\omega - \{0\})$ is an (*abstract*) *character* if for all $n > 0$ and k:

$$(k, n+1) \in K \Rightarrow (k,n) \in K$$

- For any abstract $\mathbf{\Sigma}^0_2$ character K, there is a computable abelian p-group G with character K and with $D(G)$ isomorphic to $\oplus_\omega \mathbb{Z}(p^\infty)$.

- If $A = (\omega, E)$ is a c.e. equivalence structure, then its character $\chi(A)$ is a $\mathbf{\Sigma}^0_2$ set.

- (Calvert, Cenzer, Harizanov and Morozov)
  For any abstract $\mathbf{\Sigma}^0_2$ character K, there exists a computable equivalence structure A with infinitely many infinite equivalence classes and character K.

- (Corollary)
  If A is a computably enumerable (c.e.) equivalence structure with infinitely many infinite equivalence classes, then A is isomorphic to a computable equivalence structure.

- A function $f : \omega^2 \to \omega$ is a Khisamiev's s_1-*function* if for every i and s: $f(i,s) \leq f(i,s+1)$, the limit $m_i = lim_t f(i,t)$ exists, and $m_0 < m_1 < \cdots < m_i < m_{i+1} < \cdots$

  If f is computable, the set $M = \{m_i : i \in \omega\}$ is Δ^0_2.

- If a computable equivalence structure A has an unbounded character, then there is a computable s_1-function f such that A contains an equivalence class of size m_i for each i.

- There is an infinite Δ^0_2 set D such that for any computable equivalence structure A with unbounded character and no infinite equivalence classes, $\{k : (k,1) \in \chi(A)\}$ is not a subset of D.

- For any abstract $\mathbf{\Sigma}^0_2$ character K and any finite $r \in \omega$, there is a c.e. equivalence structure with character K and with exactly r infinite equivalence classes.

- (Corollary)
  There is a c.e. equivalence structure that is not isomorphic to any computable equivalence structure.

- A structure A is *decidable* if its elementary diagram $D^e(A)$ is computable.

- A structure A is called n-*decidable*, for $n \geq 0$, if the Σ_n^0-diagram of A is computable. In particular, a structure is 0-decidable iff it is computable.

- For every n, there exists an n-decidable structure that is not $(n+1)$-decidable.

- There exists an undecidable structure that is n-decidable for all n.

- A structure A is *automorphically trivial* if there is a finite set $F \subseteq A$ such that every permutation of the domain of A that fixes F pointwise is an automorphism of A.

- (Harizanov, Knight and Morozov)

  For every automorphically nontrivial structure A, and every set $X \geq_T D^e(A)$, there exists $B \cong A$ such that

  $$D^e(B) \equiv_T D(B) \equiv_T X$$

  For every automorphically trivial structure A, we have $D^e(A) \equiv_T D(A)$.

Degree Spectrum of a Structure

- The *Turing degree spectrum* of A is
$$DgSp(A) = \{\text{deg}(B) : B \cong A\}.$$

- (Marker) For a nonstandard model A of PA, $DgSp(A)$ is closed *upward*.

- (Knight) (i) If A is automorphically nontrivial, then $DgSp(A)$ is closed *upward*.

  (ii) If A is automorphically trivial, then $|DgSp(A)| = 1$.

- If the language of A is finite, then A is automorphically trivial iff $DgSp(A) = \{\mathbf{0}\}$.

- (Hirschfeldt, Khoussainov, Shore and Slinko) For every automorphically nontrivial structure A, there is a structure B, which can be:
  a symmetric irreflexive graph,
  a partial ordering,
  a lattice,
  a ring,
  an integral domain of arbitrary characteristic,
  a commutative semigroup,
  a 2-step nilpotent group, such that

$$DgSp(A) = DgSp(B)$$

- (R. Miller, Poonen, Schoutens and Shlapentokh)
  a field

- $\mathcal{D}$ = the set of all Turing degrees
(Wehner; Slaman) There is a structure A such that

$$DgSp(A) = \mathcal{D} - \{\mathbf{0}\}$$

- (Hirschfeldt)
There is a complete decidable theory with all types computable, prime model of which has no computable copy, but has an X-decidable copy for every $X >_T \emptyset$.

- (R. Miller)
There is a linear ordering M such that $DgSp(M) \cap \mathbf{\Delta}_2^0 = \mathbf{\Delta}_2^0 - \{\mathbf{0}\}$.

- (Fokina, Harizanov and Turetsky)
If an equivalence structure A has no computable copy, then there is a perfect set of oracles that cannot compute any isomorphic copy of A.

- (Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon)
  Let $n \geq 1$.

  (1) There is a structure *A* such that

  $$DgSp(A) = \{\mathbf{c} \in \mathcal{D}\colon\ \mathbf{c}^{(n)} > \mathbf{0}^{(n)}\}$$

  A degree $\mathbf{c}$ is *nonlow*$_n$ if $\mathbf{c}^{(n)} > \mathbf{0}^{(n)}$.

  (2) There is a structure *B* such that its spectrum consists of the degrees that are *high*$_n$-or-above:

  $$DgSp(B) = \{\mathbf{c} \in \mathcal{D}\colon\ \mathbf{c}^{(n)} \geq \mathbf{0}^{(n+1)}\}$$

  A degree $\mathbf{c}$ is *high*$_n$ if $\mathbf{c}^{(n)} = \mathbf{0}^{(n+1)}$.

- (Frolov, Harizanov, Kalimullin, Kudinov and R. Miller)
  Let $n \geq 2$. Let $\mathbf{d}$ be a Turing degree.

  (1) There is a linear order L such that

  $$DgSp(L) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c}^{(n)} > \mathbf{d}\}$$

  In particular, there is a linear order the spectrum of which contains exactly the *nonlow*$_n$ degrees.
  Open for $n = 1$.

  (2) There is a linear order H such that

  $$DgSp(H) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c}^{(n)} \geq \mathbf{d}\}$$

  In particular, there is a linear order the spectrum of which contains exactly the *high*$_n$-or-above degrees.
  Not possible for $n = 1$ (Knight).

- V_∞: computable $\aleph_0$-dimensional vector space over a computable field F, which has a dependence algorithm
  $(\mathcal{L}(V_\infty), \subseteq, \cap, +)$ is the lattice of c.e. vector subspaces of V_∞

- (Guichard) There are countably many automorphisms of $\mathcal{L}(V_\infty)$. They are induced by *computable* semilinear transformations; its group denoted by GSL.

- (μ, σ) is a *semilinear* transformation if $\mu : V_\infty \to V_\infty$, σ is an automorphism of F, and for every $u, v \in V_\infty$ and $a, b \in F$:

$$\mu(au + bv) = \sigma(a)\mu(u) + \sigma(b)\mu(v)$$

- (Dimitrov, Harizanov and Morozov)

$$DgSp(GSL) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c} \geq \mathbf{0}''\}$$

- (Jockusch and Richter)
  The (*Turing*) *degree* of the *isomorphism type* of A, if it exists, is the *least* Turing degree in $DgSp(A)$.

- *Effective Extendability Condition* for A

  For every finite structure C isomorphic to a substructure of A, and every embedding f of C into A, there is an algorithm that determines whether a given finite structure D extending C can be embedded into A by an embedding extending f.

- (Richter)
  Assume that a structure A satisfies the effective extendability condition. If the degree of the isomorphism type of A exists, then it must be $\mathbf{0}$. ($DgSp(A)$ will contain a minimal pair of degrees.)

- (Richter)

  (i) A *linear ordering* without a computable copy does not have the isomorphism type degree.

  (ii) A *tree* without a computable copy does not have the isomorphism type degree.

- (A. Khisamiev)

  An *abelian p-group* without a computable copy does not have the isomorphism type degree.

- *Richter's Combination Method*

  Let T be a theory in a finite language L such that there is a computable sequence $A_0, A_1, A_2, \ldots$ of *finite* structures for L, which are *pairwise nonembeddable*. Assume that for every $X \subseteq \omega$, there is a model A_X of T such that

  $$A_X \leq_T X,$$

  and for every $i \in \omega$:

  $$A_i \text{ is embeddable in } A_X \Leftrightarrow i \in X.$$

  Then for every Turing degree $\mathbf{d}$, there is a model of T the isomorphism type of which has degree $\mathbf{d}$.

- (Richter)
  For every Turing degree $\mathbf{d}$, there is a *torsion abelian group* the isomorphism type of which has the degree $\mathbf{d}$. There is such a group the isomorphism type of which does not have a degree.

- (Dabkowska, Dabkowski, Sikora and Harizanov)
  There are various centerless groups *G* that have arbitrary Turing degrees for their isomorphism types, as well as ones with no degrees.

  G is *centerless* if $Z(G) = \{e\}$

- (Harizanov and Maeda)
  There are various structures $(M, *)$, where $*$ is not necessarily associative but is self-distributive, such that the structures have arbitrary Turing degrees for their isomorphism types, as well as no degrees.

- (Calvert, Harizanov and Shlapentokh)
  Let $\mathcal{C}$ be a class of countable structures in a finite language L, closed under isomorphism. Assume that there is a computable sequence $(A_i)_{i\in\omega}$ of computable structures in $\mathcal{C}$ satisfying the following conditions. There exists a finitely generated structure $A \in \mathcal{C}$ such that for all $i \in \omega$, we have that $A \subseteq A_i$. For any $X \subseteq \omega$, there is a structure A_X in $\mathcal{C}$ such that $A \subseteq A_X$ and $A_X \leq_T X$, and for every $i \in \omega$, there exists an embedding σ such that

  $$\sigma : A_i \hookrightarrow A_X \ \wedge \ \sigma \upharpoonright A = id$$

  iff $i \in X$. Suppose that any A_X is isomorphic to some structure B under isomorphism $\tau : A_X \simeq B$. Consider a pair of structures A_i, A_j such that exactly one of them embeds in B via σ with $(\tau^{-1} \circ \sigma) \upharpoonright A = id$. Then there is a uniform procedure with oracle B to decide which of the two structures embeds in B.
  Then for every Turing degree $\mathbf{d}$, there is a structure in $\mathcal{C}$ the isomorphism type of which has degree $\mathbf{d}$.

(Calvert, Harizanov and Shlapentokh)

- There are various algebraic fields and torsion-free abelian groups of any finite rank greater than 1, the isomorphism types of which have arbitrary Turing degrees.
  There are structures in each of these classes the isomorphism types of which do not have Turing degrees.

- Ringed spaces corresponding to unions of varieties, ringed spaces corresponding to unions of subvarieties of certain fixed varieties, and schemes over a fixed field can have arbitrary Turing degrees for their isomorphism types, as well as no degrees.

Degree Spectrum of a Relation on a Structure

- Let R be a *new* relation on computable A.
  The set of Turing degrees of images of R in *computable* isomorphic copies of A is called the *degree spectrum of* R on A:

$$DgSp(R) = \{\deg f(R) \mid f : A \cong B \ \&\ B \text{ is computable}\}$$

- *Example:* linear ordering $(L, <)$
  $Succ_L(a, b) \Leftrightarrow a < b \ \wedge \neg\exists c\ (a < c < b)$

  For a linear ordering L with only finitely many successor pairs, we have $DgSp(Succ_L) = \{\mathbf{0}\}$.

  (Downey and Moses) There is a linear ordering L_1 with $DgSp(Succ_{L_1}) = \{\mathbf{0}'\}$.

- $DgSp(Succ_{(\omega,<)}) = \{\mathbf{d} \in \mathcal{D} : \mathbf{d}$ is c.e.$\}$

- (Chubb, Frolov and Harizanov) If L is a computable linear ordering such that
$L \models (\forall x)(\exists a, b)[x < a \wedge Succ(a, b)]$,
then $DgSp(Succ_{\mathcal{L}})$ is closed upward in c.e. degrees.

- (Downey, Lempp and Wu) True for all computable linear orderings where the successor relation is infinite.

- The relation R is *intrinsically* *P* on A if in all *computable* isomorphic copies of A, the image of R is *P*.

$\{0\}$ vs. Infinite Degree Spectra

- (Hirschfeldt) A *computable* relation R on a *computable linear ordering* is either definable by a *quantifier-free* formula with parameters (hence intrinsically computable), or $DgSp(R)$ is infinite.

- (Downey, Goncharov and Hirschfeldt) A *computable* relation on a *computable Boolean algebra* is either definable by a *quantifier-free* formula with parameters, or $DgSp(R)$ is infinite.

- (Khoussainov-Shore, Goncharov, Hirschfeldt, Harizanov) There are various 2-element degree spectra of computable relations.

- Let A be a computable linear ordering of type $\omega + \omega^*$, say:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1,$$

and let R be the initial segment of type ω. R is *intrinsically* $\mathbf{\Delta}^0_2$ because of the corresponding definability of R and $\neg R$:

$$\begin{aligned} x &\in R \Leftrightarrow \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \prec x_1 \prec \cdots \prec x_n \wedge x = x_n \wedge \\ \forall y[\neg(y &\prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \cdots \wedge \neg(x_{n-1} \prec y \prec x_n)]] \end{aligned}$$

and

$$\begin{aligned} x &\notin R \Leftrightarrow \bigvee_n \exists x_0 \cdots \exists x_n [x_0 \succ x_1 \succ \cdots \succ x_n \wedge x = x_n \wedge \\ \forall y[\neg(y &\succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \cdots \wedge \neg(x_{n-1} \succ y \succ x_n)]] \end{aligned}$$

Computable (Infinitary) Formulas

- A computable Σ_0 (Π_0) formula is a finitary quantifier-free formula.
  A computable Σ_α formula, $\alpha > 0$, is a *c.e. disjunction* of formulas
  $$\exists \overline{u}\, \psi(\overline{x}, \overline{u}),$$
  where ψ is computable Π_β for some $\beta < \alpha$.
  A computable Π_α formula, $\alpha > 0$, is a *c.e. conjunction* of formulas
  $$\forall \overline{u}\, \psi(\overline{x}, \overline{u}),$$
  where ψ is computable Σ_β for some $\beta < \alpha$.

- (Ash) A relation defined in a countable structure A by a computable Σ_α (Π_α) formula is Σ^0_α (Π^0_α) relative to the atomic diagram of A.

Computability vs. Definability of Relations

- The relation R is *formally c.e.* ($\mathbf{\Sigma}^0_\alpha$) on A if R is definable by a computable $\mathbf{\Sigma_1}$ ($\mathbf{\Sigma}_\alpha$) formula with finitely many parameters.

  (Ash and Nerode) Under some effectiveness condition (enough to have the existential diagram of (A, R) computable), R is *intrinsically c.e.* on A iff R is *formally c.e.* on A. (Barker generalized this result to $\mathbf{\Sigma}^0_\alpha$.)

- R is *relatively intrinsically P* on A if in *all* isomorphic copies B of A, the image of R is P relative to the atomic diagram of B.

  (Ash-Knight-Manasse-Slaman, Chisholm) The relation R is *relatively intrinsically* $\mathbf{\Sigma}^0_\alpha$ on A iff R is *formally* $\mathbf{\Sigma}^0_\alpha$ on A. (*No* additional effectiveness needed.)

- (Goncharov, Manasse)
  There is a computable structure with an intrinsically c.e., but *not relatively* intrinsically c.e. relation.

- (Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon)
  For every computable *successor* ordinal α, there is a computable structure with a relation that is intrinsically $\mathbf{\Sigma}^0_\alpha$, but *not relatively* intrinsically $\mathbf{\Sigma}^0_\alpha$.

- (Chisholm, Fokina, Goncharov, Harizanov, Knight and Quinn)
  For every computable *limit* ordinal α, there is a computable structure with a relation that is intrinsically $\mathbf{\Sigma}^0_\alpha$, but *not relatively* intrinsically $\mathbf{\Sigma}^0_\alpha$.

Realizing All Computably Enumerable Degrees

(Harizanov)

- Under some effectiveness condition (enough to have the existential diagram of (A, R) computable), if R is *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. Turing degrees*.

At least one of R, $\neg R$ is not definable in A by a computable $\mathbf{\Sigma_1}$ formula with parameters.

- Under some effectiveness condition, if R is *intrinsically c.e.* and *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. Turing degrees*.

$\neg R$ is not definable in (A, R) by a computable $\mathbf{\Sigma_1}$ formula in which the symbol R occurs only positively.

(Ash and Knight)

- Degrees coarser than Turing degrees:

$$X \leq_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow X \leq_T Y \oplus \mathbf{\Delta}^0_\alpha$$

$$X \equiv_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow (X \leq_{\mathbf{\Delta}^0_\alpha} Y \wedge Y \leq_{\mathbf{\Delta}^0_\alpha} X)$$

$$\equiv_{\mathbf{\Delta}^0_1} \text{ is } \equiv_T$$

- Under some effectiveness conditions, if R is *not intrinsically* $\mathbf{\Delta}^0_\alpha$ on computable A, then for every $\mathbf{\Sigma}^0_\alpha$ set C, there is an isomorphism f from A onto a computable structure such that $f(R) \equiv_{\mathbf{\Delta}^0_\alpha} C$.

Not possible to replace these by Turing degrees.

Intrinsically Δ^1_1 Relations

- Suppose that A is computable, R is Δ^1_1 and invariant under automorphisms of A. Then R is definable in A by a computable formula without parameters.

- (Soskov) For R on a computable A the following are equivalent:
  (i) R is *intrinsically* Δ^1_1,
  (ii) R is *relatively intrinsically* Δ^1_1,
  (iii) R is definable in A by a computable formula with finitely many parameters.
  R is intrinsically Δ^1_1 on A
  $\Rightarrow$ R has countably many automorphic images
  $\Rightarrow$ $(\exists \vec{c})$ [R invariant under automorphisms of $(A, \vec{c})$]
  $\Rightarrow$ R definable by a computable formula $\psi(x,\vec{c})$.

Intrinsically Π^1_1 Relations

- A relation R on A is *formally* Π^1_1 if it is definable in A by a Π^1_1 disjunction of computable formulas with finitely many parameters.

  (Soskov) For a computable structure A and
  a relation R on A, the following are equivalent:
  (i) R is *intrinsically* Π^1_1,
  (ii) R is *relatively intrinsically* Π^1_1,
  (iii) R is *formally* Π^1_1.

- A *Harrison ordering* A is a *computable* ordering of type $\omega_1^{CK}(1+\eta)$.

  R^A, the initial segment of type ω_1^{CK}, is *intrinsically* Π^1_1 since
  it is defined by the disjunction of computable formulas
  saying that the interval to the left of x has ordering type α,
  for computable ordinals α.

- A *Harrison group* is a computable abelian p-group H with length ω_1^{CK}, and Ulm invariants $u_H(\alpha) = \infty$ for all computable α, and with infinite dimensional divisible part.

  R^H, the set of elements that have computable ordinal height (the complement of the divisible part), is intrinsically Π^1_1 since it is defined by the disjunction of computable formulas saying that x has height α, for computable α.

- (Goncharov, Harizanov, Knight and Shore)

  The following sets are equal:

  (i) the set of Turing degrees of maximal well-ordered initial segments of Harrison orderings;

  (ii) the set of Turing degrees of left-most paths of computable subtrees of $\omega^{<\omega}$ in which there is a path but not a hyperarithmetic one;

  (iii) the set of Turing degrees of Π^1_1 paths through Kleene's $\mathcal{O}$;

  (iv) the set of Turing degrees of the height-possessing parts of Harrison groups.

Unbounded Degree Spectra of Relations

- (Kueker) The following are equivalent for countable A:
  (i) R has fewer than $2^{\aleph_0}$ different images under automorphisms of A;
  (ii) R is definable in A by an $L_{\omega_1\omega}$ formula with finitely many parameters.

- (Harizanov) There is an uncountable degree spectrum of a computable relation on a computable structure, which consists of $\mathbf{0}$ and pairwise incomparable nonzero Turing degrees.

- (Ash-Cholak-Knight, Harizanov) For a computable relation R on computable A, if $DgSp(R)$ contains *every* Δ_3^0 *Turing degree*, obtained via an isomorphism f of the same Turing degree as $f(R)$, then $DgSp(R) = \mathcal{D}$.

Spectrally Universal Models

- (Harizanov and R. Miller)
  For any countable linear ordering A, there is a unary relation R on $\mathcal{Q} = (\mathbb{Q}, <)$ such that $DgSp(A) = DgSp_{\mathcal{Q}}(R)$.

  A structure $\mathcal{U}$ is said to be *spectrally universal* for a theory T if for every automorphically nontrivial countable model A of T, there is an embedding $f : A \rightarrow \mathcal{U}$ such that A as a structure, has the same degree spectrum as $f(A)$ as a relation on $\mathcal{U}$.

  Countable dense linear ordering and the random graph are spectrally universal.

- (Csima, Harizanov, R. Miller and Montalbán)
  The countable atomless Boolean algebra is spectrally universal.

THANK YOU!