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Computable Structure Theory, Part II

Complexity of Isomorphisms and Automorphisms

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Computably and Relatively Computably Categorical Structures

Let A be a *computable* structure.

- A is *computably categorical* if for all computable $B \cong A$, there is a computable isomorphism f from A onto B.

- A is *relatively computably categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is computable relative to the atomic diagram of B.

- A is relatively computably categorical $\Rightarrow$ A is computably categorical

Examples

- $(\mathbb{Q}, <)$ is computably categorical (back-and-forth argument).

  Not every isomorphism from $(\mathbb{Q}, <)$ to a computable structure is computable (not *computably stable*).

  $(\omega, <)$ is not computably categorical.

  The field $\mathbb{Q}$ is computably categorical.

- (R. Miller and Shoutens)
  There is a computable computably categorical field of *infinite* transcendence degree.

Linear Orderings

- (Goncharov-Dzgoev, Remmel)
  A computable linear ordering A is *computably categorical iff*
  A has only finitely many successor pairs *iff*
  A is relatively computably categorical.

Boolean Algebras

- (LaRoche, Goncharov-Dzgoev, Remmel)
  A computable Boolean algebra B is *computably categorical iff*
  B has finitely many atoms *iff*
  B is relatively computably categorical.

Algebraic Fields with Splitting Algorithm

- (R.Miller and Shlapentokh)
  A computable algebraic field F with a splitting algorithm is *computably categorical iff* the *orbit relation*, $\{(a, b) \in F^2 : (\exists h \in Aut(F))[h(a) = b]\}$, is decidable *iff* F is relatively computably categorical.

- F has a *splitting algorithm* if it is decidable which polynomials in $F[x]$ are irreducible.

Abelian p-groups

- (Goncharov, Smith)

  A computable abelian p-group *G* is *computably categorical iff* *G* is isomorphic to:

  (1) $\bigoplus_{\alpha} \mathbb{Z}(p^{\infty}) \oplus H$, where $\alpha \leq \omega$ and H is finite, *or*

  (2) $\bigoplus_{n} \mathbb{Z}(p^{\infty}) \oplus H \oplus \bigoplus_{\omega} \mathbb{Z}(p^{k})$, where $n, k \in \omega$ and H is finite.

  A computable computably categorical abelian p-group is relatively computably categorical.

Equivalence Structures

- (Calvert, Cenzer, Harizanov and Morozov)

  A computable equivalence structure $A = (\omega, E)$ is
  computably categorical iff either
  (1) A finitely many finite equivalence classes, or
  (2) A has finitely many infinite classes, bound on the size of classes,
  and exactly one finite k with infinitely many classes of size k.

- Every computable computably categorical equivalence structure
  is *relatively* computably categorical.

Trees

- (R. Miller)
  No computable tree $(T, \prec)$ of infinite height is computably categorical.

- (Lempp, McCoy, R. Miller and Solomon)
  Every computable computably categorical tree of finite height is relatively computably categorical.

Injection Structures (Directed Graphs)

- $A = (\omega, f)$, where $f : \omega \longrightarrow \omega$ is a $1 - 1$ function.
  For $a \in \omega$, the *orbit of* a (under f) is:

$$\mathcal{O}_f(a) = \{b \in \omega : (\exists n \in \mathbb{N})[f^n(a) = b \ \vee \ f^n(b) = a]\}$$

- Two types of *infinite orbits*: $\mathbb{Z}$-orbits, ω-orbits

- (Cenzer, Harizanov and Remmel)

  A computable injection structure A is *computably categorical iff*
  A has finitely many infinite orbits *iff*
  A is relatively computably categorical.

Computably Categorical but Not Relatively

- (Goncharov)
  There is a computable structure (in fact, a rigid graph) that is computably categorical, but *not* relatively computably categorical.

- (Hirschfeldt, Khoussainov, Shore and Slinko)
  There are computable computably categorical, but *not* relatively computably categorical: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, integral domains.

- (Hirschfeldt, Kramer, Miller and Shlapentokh)
  There is a computable computably categorical algebraic field, which is *not* relatively computably categorical.

Syntactic Approach to Categoricity

- There is a syntactic condition that implies computable categoricity, and is equivalent to relative computable categoricity.
  The condition involves the existence of a certain Scott family.

- Let A be a *countable* structure.

  (*Scott Isomorphism Theorem*) There is an $L_{\omega_1\omega}$ sentence, called Scott sentence for A, the countable models of which are exactly the isomorphic copies of A.

- Scott sentence for A is derived from a family of $L_{\omega_1\omega}$ formulas defining the orbits (under automorphisms) of tuples in A.

Scott Families

Let A be a *countable* structure.

- A *Scott family for* A is a set Φ of $L_{\omega_1\omega}$ formulas, with a fixed finite tuple of parameters $\overline{c}$ in A, such that:

1. Each tuple $\overline{a}$ in A satisfies some $\psi(\overline{c}, \overline{x}) \in \Phi$, and

2. If $\overline{a}$, $\overline{b}$ are tuples in A (of the same length) satisfying the *same* formula $\psi(\overline{c}, \overline{x}) \in \Phi$, then there is an *automorphism* of A taking $\overline{a}$ to $\overline{b}$.

- (Goncharov)

  A computable structure A is *relatively* computably categorical *iff* A has a c.e. Scott family of (finitary) existential formulas.

- (Cholak, Shore and Solomon)

  There is a computably categorical (rigid) structure with *no* Scott family of *finitary* formulas.

Back-and-Forth Construction

- If a computable structure A has a c.e. Scott family Φ of existential formulas, then A is *relatively* computably categorical.

- *Proof sketch*
  $\overline{c}$ parameters; let $(A, \overline{c}) \cong (B, \overline{d})$.
  Will construct an isomorphism f computable in the atomic diagram of B.
  $f = \bigcup_s f_s, \;\; f_s \subseteq f_{s+1}$.
  Assume f_s maps $\overline{c}\hat{\ }\overline{a} \to \overline{d}\hat{\ }\overline{b}$,
  $q \in A$, where $q \notin \overline{c}\hat{\ }\overline{a}$.
  Find $\psi(\overline{c}, \overline{x}, y) \in \Phi$ and $r \in B$ such that

$$A \models \psi(\overline{c}, \overline{a}, q) \;\wedge\; B \models \psi(\overline{d}, \overline{b}, r).$$

  Let $f_{s+1}(q) = r$.

- A computable structure A is *relatively* computably categorical *iff* A has a c.e. Scott family of computable $\mathbf{\Sigma_1}$ formulas.

- Computable $\mathbf{\Sigma_1}$ formula: $\bigvee_{i \in I} \exists \overline{u_i}\, \theta_i(\overline{x}, \overline{u_i})$,
  where I is c.e. and θ_i's quantifier-free.

Δ^0_α-Categoricity and Relative Δ^0_α-Categoricity

- Let α be a computable ordinal, and A a computable structure.

  (i) A is Δ^0_α-*categorical* if for all computable $B \cong A$,
  there is a Δ^0_α isomorphism f from A onto B.

  (ii) A is *relatively* Δ^0_α-*categorical* if for all $B \cong A$,
  there is an isomorphism f from A onto B,
  which is Δ^0_α relative to the atomic diagram of B.

- Δ^0_1-categorical: computably categorical.
  Δ^0_2-categorical: limit computably categorical or $\mathbf{0}'$-categorical,
  where $\mathbf{0}'$ is the Turing degree of the halting set.

- $\mathcal{A}$ is relatively Δ^0_α-categorical $\Rightarrow$ $\mathcal{A}$ is Δ^0_α-categorical

Equivalence of Semantic and Syntactic Conditions

- (Ash-Knight-Manasse-Slaman, Chisholm)

  A computable structure A is *relatively* $\boldsymbol{\Delta}^0_\alpha$-categorical *iff*

  A has a c.e. Scott family of computable $\boldsymbol{\Sigma}_\alpha$ formulas *iff*

  A has a $\boldsymbol{\Sigma}^0_\alpha$ Scott family of computable $\boldsymbol{\Sigma}_\alpha$ formulas
  (*A* has a *formally* $\boldsymbol{\Sigma}^0_\alpha$ Scott family)

- Computable $\boldsymbol{\Sigma}_2$ formula:

$$\bigvee_{j \in J} \exists \overline{u}_j \bigwedge_{i \in I} \forall \overline{v}_{ij} \theta_{ij}(\overline{x}, \overline{u}_j, \overline{v}_{ij}),$$

  where I and J are c.e. and θ_{ij}'s are quantifier-free.

Examples

- $(\omega, <)$ is relatively Δ^0_2-categorical.
  Any computable equivalence structure is relatively Δ^0_3-categorical.
  Any computable injection structure is relatively Δ^0_3-categorical.

- (Lempp, McCoy, Miller and Solomon)
  For every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1}-categorical but not Δ^0_n-categorical.

- (Calvert, Harizanov, Knight and Quinn)
  Assume that G is a computable reduced Abelian p-group.
  If $\lambda(G) = \omega \cdot n$, then G is not Δ^0_{2n-1}-categorical.

Extra Decidability on Categorical Structures

- (Goncharov)
  Assume that A is 2-*decidable*. If A is computably categorical, then A is relatively computably categorical.

  A is n-*decidable* if $\mathbf{\Sigma}_n^0$ diagram of A is computable.

- (Ash)
  Let $\alpha > 1$ be a computable ordinal.
  Under some additional decidability on A, if A is $\mathbf{\Delta}_\alpha^0$-categorical, then A is relatively $\mathbf{\Delta}_\alpha^0$-categorical.

- (Kudinov)
  There is a 1-*decidable* structure that is computably categorical, but *not* relatively computably categorical.

Fraïssé limits

- The *age* of a structure A is the class of all finitely generated structures that can be embedded in A.

- A structure A is *ultrahomogeneous* if every isomorphism between finitely generated substructures of A extends to an automorphism of A.

- A structure A is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if A is countable, ultrahomogeneous, and has age $\mathbb{K}$. (A structure A is a Fraïssé limit if for some class $\mathbb{K}$, A is the Fraïssé limit of $\mathbb{K}$.)

- (Fraïssé) Fraïssé limit of a class of finitely generated structures is unique up to isomorphism.

- (Fokina, Harizanov and Turetsky)
  There is a 1-decidable structure M that is a Fraïssé limit and is computably categorical, but not relatively computably categorical. Moreover, the language for such M can be finite or relational.

- Let A be a computable structure which is a Fraïssé limit. Then A is relatively $\mathbf{\Delta}^0_2$-categorical.

- If the language of a Fraïssé limit A is finite and relational, then A is relatively computably categorical.

- (Cholak, Goncharov, Khoussainov and Shore)
  There is a computable computably categorical structure A such that for every $a \in A$, the expanded structure (A, a) is *not* computably categorical.

- (T. Millar)
  If a computably categorical structure A is 1-decidable, then any expansion of A by finitely many constants remains computably categorical.

- (Downey, Kach, Lempp and Turetsky)
  Any 1-decidable computably categorical structure is relatively Δ^0_2-categorical.

- (Downey, Kach, Lempp, Lewis, Montalbán and Turetsky)
  For every computable ordinal α, there is a computably categorical structure that is *not* relatively Δ^0_α-categorical.

Non-Relatively Δ^0_2-Categorical Tree

- (Fokina, Harizanov and Turetsky)
  There is a Δ^0_2-categorical tree of finite height,
  which is not relatively Δ^0_2-categorical.

- *Proof sketch.* Build a computable tree $\mathcal{T}$.
  Diagonalize against all potential Scott families: consider all pairs $(\mathcal{X}, \overline{c})$, where $\mathcal{X}$ is a c.e. family of computable Σ_2 formulas and $\overline{c}$ is a finite tuple of elements from the domain of $\mathcal{T}$.
  Assure that every isomorphic computable tree is $\mathbf{0}'$-isomorphic to $\mathcal{T}$.

- *Open Problems:* Characterize computable relatively Δ^0_2-categorical trees of finite height.
  Characterize computable Δ^0_2-categorical trees of finite height.

Relatively Δ^0_2-Categorical Equivalence Structures

- (Calvert, Cenzer, Harizanov and Morozov)
  A computable equivalence structure A is *relatively* Δ^0_2*-categorical iff*:
  (1) A has finitely many infinite equivalence classes, or
  (2) A has bounded character.

- *Open Problem:* Characterize computable Δ^0_2-*categorical* equivalence structures.

Khisamiev Functions

- A function $f : \omega^2 \rightarrow \omega$ is a Khisamiev s-*function* if for every i and t:
  (i) $f(i,t) \leq f(i,t+1)$, and the limit $m_i = \lim_{l \to \infty} f(i,l)$ exists.
  f is called Khisamiev s_1-*function* if, in addition:
  (ii) $m_0 < m_1 < \cdots < m_i < m_{i+1} < \cdots$

- Let A be a computable equivalence structure with *finitely* many infinite equivalence classes and infinite character $\chi(A)$.

  There exists a computable Khisamiev s-function with limits m_i such that:

  $$(k,n) \in \chi(A) \Leftrightarrow card(\{i : k = m_i\}) \geq n$$

  If $\chi(A)$ is unbounded, then there is a computable Khisamiev s_1-function such that A contains an equivalence class of size m_i for every i.

- (Calvert, Cenzer, Harizanov and Morozov)
  Let A be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character, which has a computable Khisamiev s_1-function. Then A is *not* Δ^0_2-categorical.

Non-Relatively Δ^0_2-Categorical Equivalence Structure

- (Kach and Turetsky)
  There is a computable Δ^0_2-categorical equivalence structure M, which is *not* relatively Δ^0_2-categorical.

  M has infinitely many infinite equivalence classes
  and unbounded character,
  but has no computable Khisamiev s_1-function,
  and has only finitely many equivalence classes of size k for any finite k.

Relatively Δ^0_2-Categorical Abelian p-Groups

- The *period* of a group *H* is $max\{o(h) : h \in H\}$ if finite, and ∞ otherwise.

- (Calvert, Cenzer, Harizanov and Morozov)
  A computable abelian p-group *G* is *relatively* Δ^0_2*-categorical iff*

  (1) *G* is isomorphic to $\bigoplus_\alpha \mathbb{Z}(p^\infty) \oplus H$, where $\alpha \leq \omega$ and *H* has finite period; *or*

  (2) all elements in *G* are of finite height (equivalently, reduced with $\lambda(G) \leq \omega$).

- A computable equivalence structure A is relatively Δ^0_2-categorical *iff*:
  (1) A has finitely many infinite equivalence classes, or
  (2) A has bounded character.

- Even for a group G of infinite period with only finitely many $\mathbb{Z}(p^\infty)$ components, G is not relatively Δ^0_2-categorical.

  This differs from equivalence structures, where each equivalence class is necessarily computable, but $D(G)$ need not be computable even when there is just one copy of $\mathbb{Z}(p^\infty)$.

- *Open Problem*: Characterize computable Δ^0_2-categorical abelian p-groups.

- (Calvert, Cenzer, Harizanov and Morozov)
  Let *A* be a computable group isomorphic to $\bigoplus_{\alpha} \mathbb{Z}(p^{\infty}) \oplus H$, where all elements of *H* are of finite height. Then *A* is relatively Δ^0_3-categorical.

- (Cenzer, Harizanov and Remmel)
  A computable injection structure A is Δ^0_2-categorical *iff*
  A has finitely many orbits of type ω or finitely many orbits of type Z

  Every computable Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical.

Non-Relatively Δ^0_2-Categorical Torsion-Free Abelian Groups

- A *homogenous, completely decomposable,* abelian group G is a group of the form $\bigoplus_{i \in I} H$, where H is a subgroup of $(\mathbb{Q}, +)$.

- G is *computably categorical iff* G is of finite rank.

- (Fokina, Harizanov and Turetsky)
  There is a computable, homogenous, completely decomposable, abelian group, which is Δ^0_2-categorical, but not relatively Δ^0_2-categorical.

- For P, a set of primes, $Q^{(P)}$ is the subgroup $(\mathbb{Q}, +)$ generated by $\{\frac{1}{p^k} \;:\; p \in P \wedge k \in \omega\}$.

- (Downey and Melnikov)
  A computable, homogenous, completely decomposable, abelian group G of infinite rank is Δ^0_2-*categorical iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $- P$) is semi-low.

- A set $S \subseteq \omega$ is *semi-low* if the set
  $H_S = \{e : dom(\varphi_e) \cap S \neq \emptyset\}$ is computable from $\emptyset'$.

  $(\varphi_e)_{e\in\omega}$ is a computable enumeration of all unary partial computable functions.

- (Fokina, Harizanov and Turetsky)
  A computable, homogenous completely decomposable abelian group G of infinite rank is *relatively* $\mathbf{\Delta}^0_2$*-categorical iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$ where P is a *computable* set of primes.

Relatively Δ^0_2-Categorical Linear Orders

- (McCoy)
  A computable linear order A is relatively Δ^0_2-categorical *iff*
  A is a sum of finitely many intervals, each of type

  $$m, \omega, \omega^*, \mathbb{Z}, \text{ or } n \cdot \eta,$$

  so that each interval of type $n \cdot \eta$ has a supremum and an infimum.

- *Open Problems:* Characterize computable Δ^0_2-*categorical* linear orders.

  Is there is a computable Δ^0_2-categorical linear order,
  which is *not* relatively Δ^0_2-categorical?

Relatively Δ^0_2-Categorical Boolean Algebras

- (McCoy)
  A computable Boolean algebra $\mathcal{B}$ is *relatively* Δ^0_2*-categorical iff* $\mathcal{B}$ can be expressed as a finite direct sum of subalgebras

  $$\mathcal{C}_0 \oplus \cdots \oplus \mathcal{C}_k$$

  where each $\mathcal{C}_k$ is either atomless, an atom, or a 1-atom.

- (Bazhenov; Harris)
  Every computable Δ^0_2-categorical Boolean algebra is relatively Δ^0_2-categorical.

Automorphism Degree Spectra

- Let A be a computable structure. The *automorphism spectrum of* A is the set of Turing degrees

$$\mathrm{AutSp}(A) = \{\mathbf{deg}\, f : f \in Aut(A)\ \&\ (\exists x \in A)(f(x) \neq x)\}$$

- There exist permutations f_0, f_1 of ω such that $f_0, f_1 \leq_T \emptyset'$ and the Turing degrees of $f_0 f_1$ and $f_1 f_0$ are incomparable.

- $\mathrm{AutSp}(A)$ is at most countable iff it contains only hyperarithmetic degrees.

Singleton Automorphism Spectra

- (Jockusch and McLaughlin) There exists an arithmetical Turing degree $\boldsymbol{d}$ such that no computable structure has automorphism spectrum $\{\boldsymbol{d}\}$.

- There exists a computable structure C such that for every c.e. degree $\boldsymbol{d}$, some computable copy of C has automorphism spectrum $\{\boldsymbol{d}\}$.

- For every $\boldsymbol{\Sigma}^0_{n+1}$ degree $\boldsymbol{d} \geq_T \mathbf{0}^{(n)}$, some computable structure has automorphism spectrum $\{\boldsymbol{d}\}$ and its isomorphism type depends only on n.

- (Odifreddi) For any Turing degrees $\boldsymbol{d}$ such that $\mathbf{0}^{(\alpha)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(\alpha+1)}$ for some computable ordinal α, there exists a computable A with automorphism spectrum $\{\boldsymbol{d}\}$.

Automorphism Spectra of Incomparable Degrees

- Let $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ be incomparable Turing degrees. Then *no* computable structure M has $\text{AutSp}(M) = \{\boldsymbol{d}_0, \boldsymbol{d}_1\}$, and *no* computable structure M_0 has $\text{AutSp}(M_0) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1\}$.

- There exist pairwise incomparable $\boldsymbol{\Delta}_2^0$ Turing degrees $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and computable structures A and B such that $\text{AutSp}(A) = \{\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$ and $\text{AutSp}(B) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$.

  There exist c.e. sets X and Y such that $X \subset Y$ and the degrees $\deg X$, $\deg(Y - X)$, $\deg Y$ are pairwise incomparable.

- If $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ is a set of Turing degrees such that each singleton $\{\boldsymbol{d}_i\}$ is an automorphism spectrum, then there exists a computable structure A the automorphism spectrum of which is the closure of $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ under joins.

- A total function $f : \omega \to \omega$ is a Π^0_1-*function singleton* if there exists a computable tree $\mathcal{T} \subseteq \omega^{<\omega}$ through which f is the *unique* infinite path.

- For a Turing degree $\boldsymbol{d}$, the following are equivalent.
(i) $\{\boldsymbol{d}\}$ is the automorphism spectrum of some computable structure A;
(ii) $\boldsymbol{d}$ contains a Π^0_1-function singleton.

- For a computable structure A, the following are equivalent:
  (i) AutSp(A) is at most countable;
  (ii) Every degree in AutSp(A) contains a Π^0_1-function singleton.

- There exists a computable structure A the spectrum of which is the *union of the upper cones* above each of an infinite antichain of c.e. degrees.

  The same holds for any finite antichain of degrees of Π^0_1-function singletons.

THANK YOU!