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Computable Structure Theory, Part III

Relations on a Structure. Descriptions of Structures

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Orders on Groups and Other Magmas

- Magma is a nonempty set with a binary operation: $(M, \cdot)$

- A *left order* on the structure $(M, \cdot)$ is
a linear ordering $\prec$ of the domain M, which
is left-invariant with respect to $\cdot$
$(\forall x, y, z)[x \prec y \Rightarrow z \cdot x \prec z \cdot y]$

- $\prec$ is a *bi-order* (*order*) on the structure if
$(\forall x, y, z)[x \prec y \Rightarrow (z \cdot x \prec z \cdot y \wedge x \cdot z \prec y \cdot z)]$

- A *partial left order* is
  a partial order $\prec$ of the domain M such that:

  $(\forall x, y, z)[x \prec y \Rightarrow (z \cdot x,\ z \cdot y \text{ are comparable} \wedge z \cdot x \prec z \cdot y)]$

- $LO(M)$ the set of all left orders on M
  $RO(M)$ the set of all right orders on M
  $BiO(M)$ the set of all bi-orders on M

- Given a left order $\prec_l$ on a group G, we have a right order $\prec_r$ on G:
  $x \prec_r y \Leftrightarrow_{def} y^{-1} \prec_l x^{-1}$

  $x \prec_r y \Rightarrow y^{-1} \prec_l x^{-1} \Rightarrow z^{-1}y^{-1} \prec_l z^{-1}x^{-1}$
  $\Rightarrow (yz)^{-1} \prec_l (xz)^{-1} \Rightarrow xz \prec_r yz$

- If G is computable, then $\prec_r$ and $\prec_l$ have the same Turing degree.

- G is left-orderable group $\Rightarrow$ G is *torsion-free*

  torsion-free: $(\forall x \in G - \{e\})[order(x) = \infty]$

  $e \prec x \Rightarrow x \prec x^2 \prec x^3 \prec \cdots \prec x^n$

- (Levy)
  G is abelian and torsion-free $\Rightarrow G$ is orderable

- (Kokorin and Kopytov)
  Every torsion-free nilpotent group is orderable.

- Torsion-free, but not left-orderable group:

  $G = \langle x, y \mid y^2xy^2 = x,\, x^2yx^2 = y \rangle$

- $(\mathbb{Z}, +)$ has two orders, both computable.

  $(\mathbb{Z}^2, +)$ has $2^{\aleph_0}$ orders.

  $\mathbb{Z}^\omega = \bigoplus_{i \in \omega} \mathbb{Z}$, the direct sum of ω copies of $\mathbb{Z}$

  $(\mathbb{Z}^\omega, +)$ has $2^{\aleph_0}$ orders.

- (Solomon)

  (i) A computable torsion-free abelian group of finite rank $n > 1$ has an order in every Turing degree.

  (ii) A computable torsion-free abelian group of infinite rank has an order in every Turing degree $\geq \mathbf{0'}$.

  (iii) A computable torsion-free properly n-step nilpotent group G has an order in every Turing degree $\geq \mathbf{0^{(n)}}$.

- (Downey and Kurtz)

  There is a computable torsion-free abelian group G (hence orderable) such that G has no computable order.

- (Dobrica)

  Every computable torsion-free abelian group is isomorphic to a computable group with a computable basis (hence with a computable order).

- (Harrison-Trainor)

  There is a computable left-orderable group that is not isomorphic to a computable group with a computable left order.

  Not known for the case of bi-orderable groups.

- For groups, orders are often identified with their positive cones.

  Let $\prec$ be a partial left order on a group G.

  Positive cone: $P = \{a \in G : a \succeq e\}$
  Negative cone: $P^{-1} = \{a \in G : a \preceq e\}$

  1. $PP \subseteq P$ (P *sub-semigroup* of G)
  2. $P \cap P^{-1} = \{e\}$

  P with 1 & 2 defines a partial left order $\preceq_P$ on G:

$$x \preceq_P y \Leftrightarrow x^{-1}y \in P$$

  $x \preceq_P y \Rightarrow x^{-1}y \in P$

  $\Rightarrow x^{-1}z^{-1}zy = (zx)^{-1}(zy) \in P$

  $\Rightarrow zx \preceq_P zy$

- P with 1 & 2 defines a *left order* iff

  3. $P \cup P^{-1} = G$

- P with 1, 2 & 3 defines a *bi-order* iff:
  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ (P *normal*)

  bi-order $\prec$; let $g \in G$
  $y \succ e \Leftrightarrow g^{-1}y \succ g^{-1} \Leftrightarrow g^{-1}yg \succ e$

  P normal; let $x \preceq_P y$, $z \in G$

  right invariant: $x^{-1}y \in P \Rightarrow z^{-1}x^{-1}yz \in P$
  $(xz)^{-1}yz \in P \Rightarrow xz \preceq_P yz$

- Example: $G = \mathbb{Z} \oplus \mathbb{Z}$ has a bi-order with positive cone

$$P = \{(a,b) \mid a > 0 \vee (a = 0 \wedge b \geq 0)\}.$$

- Fundamental group of Klein bottle
  $K = \left\langle a, b \mid a^{-1}ba = b^{-1} \right\rangle$ left-orderable, but not bi-orderable.
  $ba = ab^{-1}$

  Positive cone $P = \{a^n b^m \mid n > 0 \vee (n = 0 \wedge m \geq 0)\}$
  defines a left order on G.

  Not bi-orderable:
  $b \succ e \Rightarrow a^{-1}ba = b^{-1} \succ e$

- (Solomon)
  For every bi-orderable computable group G, there is
  a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection
  from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- Hence, by the *Low Basis Theorem* of Jockusch and Soare,
  $BiO(G)$ contains a low order.
  Similarly, $LO(G)$ contains a *low* order.
  A set X and its Turing degree $\mathbf{x}$ are called *low* if $X' \leq_T \emptyset'$.

- A bi-orderable (left-orderable) computable group has a bi-order
  (left order) of c.e. degree.

- If a computable group has only finitely many bi-orders (left orders),
  they are all computable.

- A set A is *truth-table reducible* to a set B: $A \leq_{\text{tt}} B$
  if there is a computable function h and an index e so that

$$A(x) = \varphi_e^{B \upharpoonright h(x)}(x).$$

  and for any string $\sigma \in 2^{<\omega}$ of length $h(x)$, we have $\varphi_e^{\sigma}(x) \downarrow$.

- $(\varphi_e)_{e \in \omega}$ a computable enumeration of all unary partial computable functions
  A set $X \leq_T \emptyset'$ is *super low* if $X' \leq_{\text{tt}} \emptyset'$.

- Hence, by the *Super Low Basis Theorem* of Jockusch and Soare,
  a computable bi-orderable (left-orderable) group contains
  a *super low* order.

- For a bi-order $\prec$: $(a \prec b \wedge c \prec d \Rightarrow a \cdot c \prec b \cdot d)$
  $a \prec b \Rightarrow a \cdot c \prec b \cdot c$
  $c \prec d \Rightarrow b \cdot c \prec b \cdot d$

- Not necessarily true for a left order.

  Example: Klein bottle group $K = \left\langle a, b \mid a^{-1}ba = b^{-1} \right\rangle$

  $ba = ab^{-1}$

  b and a^2 commute:
  $a^2b = a^2(a^{-1}b^{-1}a) = ab^{-1}a = ba^2$

  $ba \neq a$ but $(ba)^2 = baba = ab^{-1}ba = a^2$

- A magma $(Q, *)$ is called a *quandle* if:

  1. $(\forall a)[a * a = a]$ (idempotence);

  2. for every $b \in Q$, the mapping $*_b : Q \to Q$ defined by

  $$*_b(a) = a * b$$

  is bijective;

  3. $(\forall a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (right self-distributivity).

- A quandle Q is called *trivial* if the operation $*$ is defined by

  $$(\forall a, b)[a * b = a].$$

  Every linear ordering of elements of trivial Q is right-invariant.

- For a group G, the *conjugate* quandle Conj(G) is one with domain G and the operation $*$ given by $a * b = b^{-1}ab$.

- 1. $a * a = a^{-1}aa = a$

  2. $\forall b \; \forall c \; \exists ! a \; [a * b = c]$

  $b^{-1}ab = c \Rightarrow a = bcb^{-1}$

  3. $(a * b) * c = c^{-1}(a * b)c = c^{-1}b^{-1}abc$

  $(a * c) * (b * c) = (c^{-1}ac) * (c^{-1}bc) = (c^{-1}bc)^{-1}(c^{-1}ac)(c^{-1}bc)$
  $= c^{-1}b^{-1}cc^{-1}abc = c^{-1}b^{-1}abc$

  Then every bi-order on G induces a right order on Conj(G).

- Let B be a bi-order on G. Define R on Conj(G) as

$$(\forall a, b)[(a, b) \in R \Leftrightarrow (a, b) \in B]$$

  R is right-invariant on Conj(G) because for $(a, b) \in R$ and $c \in$ Conj(G):

$$(a, b) \in B \Rightarrow (c^{-1}ac, c^{-1}bc) \in B \Rightarrow (a * c, b * c) \in R$$

- Not all right orders on Conj(G) are induced by bi-orders on G.
  It is possible to have $BiO(G) = \emptyset$,
  while $RO(\text{Conj}(G)) \neq \emptyset$.

  Let G be an abelian group with torsion.
  Then $BiO(G) = \emptyset$, but Conj(G) is a trivial quandle,
  so it admits many right orders.

- *Topology* defined on $LO(M)$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\Delta}$ where $\Delta = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(M) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki and Veve)

  Let M be a left-orderable magma with cardinality $|M| = \mathfrak{m} \geq \aleph_0$. Then $LO(M)$ is a compact space. $BiO(M)$ is also a compact space.

  By Vedenissoff's theorem, $LO(M)$ can be homeomorphically embedded into the Cantor cube $\{0,1\}^{\mathfrak{m}}$. Moreover, $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use Conrad's theorem to establish that $LO(G)$ is compact.

- (Conrad)
  A partial left order given by its positive cone P can be extended to a total left order on G iff for every $\{x_1, \ldots, x_n\} \subseteq G \backslash \{e\}$ there are $\epsilon_1, \ldots, \epsilon_n$, $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}),$$

  where $sgr(A)$ is the sub-semigroup of G generated by A.

- For a countable group G, $LO(G) \neq \emptyset$ is homeomorphic to the Cantor set iff for any sequence $(a_0, b_0), \ldots, (a_{k-1}, b_{k-1})$, $S_{(a_0,b_0)} \cap \cdots \cap S_{(a_{k-1},b_{k-1})}$ is either empty or infinite.

- (Sikora)
  The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Dabkowska)
  The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- There are countable groups with infinitely countably many bi-orders.

- (Linnell)
  The space of left orders of a countable left-orderable group is either finite or contains a homeomorphic copy of the Cantor set.

- Let $F_n = \langle x_0, x_1, \ldots, x_{n-1} \mid \ \rangle$ be a free group of rank n.

- *Conjecture* (Sikora)
  For $n > 1$, the spaces $LO(F_n)$ and $BiO(F_n)$ are homeomorphic to the Cantor set.

- (Navas-Flores)
  The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

  Not known for $BiO(F_n)$ for $n > 1$.

- (Chubb, Dabkowski and Harizanov)
  For a computable group G isomorphic to a free group F_n of rank $n > 1$, we have a bi-order in every tt-degree.

Proof sketch for Turing degrees:

For a group G, the *lower central series* is the descending sequence of subgroups $(\gamma_\alpha(G))_\alpha$ defined as:

$$\begin{aligned}
&\gamma_1(G) = G,\\
&\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G],\\
&\gamma_\beta(G) = \textstyle\bigcap_{\alpha<\beta}\ \gamma_\alpha(G), \text{ when } \beta \text{ is a limit ordinal,}
\end{aligned}$$

where $[A, B]$ is the subgroup of G generated by the elements $a^{-1}b^{-1}ab$, with $a \in A$ and $b \in B$.

- *Lower central series* of F_n: $\gamma_1(F_n) \geq \cdots \geq \gamma_i(F_n) \geq \cdots$

- (Magnus) $\gamma_\omega = \bigcap_{i=1}^{\omega} \gamma_i(F_n) = \{e\}$ residually nilpotent

- (Hall) $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$,
  where $k_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})n^d$, and μ is Möbius function

- Isomorphism uniformly computable since a basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ can be found algorithmically in n, i.

- Construct bi-orders on F_n using bi-orders on $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.

- Different choices of orders on quotients induce different orders on F_n.

- Produce a bi-order on F_n of a given Turing degree.

- (Chubb, Dabkowski and Harizanov)
  Let G be a computable group, and $\mathbb{P}$ a computable family of finite subsets of $G - \{e\}$ satisfying the following conditions for every $p \in \mathbb{P}$:

  $e \notin S(p)$

  (*branching*) $(\exists q, r \in \mathbb{P})(\exists a \in G)[(q, r \supseteq p) \wedge (a \in q) \wedge (a^{-1} \in r)]$

  (*extendability*) $(\forall a \in G - \{e\})(\exists q \in \mathbb{P})[(q \supseteq p) \wedge [(a \in q) \vee (a^{-1} \in q)]]$

  Then there is a bi-order on G in every tt-degree.

- $S(p)$ is the *normal* sub-semigroup of G generated by p.

- We can generalize the construction for free groups of finite rank > 1 to a class of finitely presented, *residually nilpotent* groups that are not nilpotent.

- (Chubb, Dabkowski and Harizanov)
Let G be a finitely presented, torsion-free, computable group.
Let $G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$ be the lower central series of G.
If $\gamma_\omega(G) = \{e\}$, and
$\gamma_i(G)/\gamma_{i+1}(G)$ is nontrivial and torsion-free for each $i = 1, 2, \ldots,$
then there is a bi-order on G in every tt-degree.

- *Examples*:

  (1) Surface groups of genus $n > 1$:
  $\langle x_1, y_1, \ldots, x_n, y_n \mid [x_1, y_1] \cdots [x_n, y_n] \rangle$

(2) Finitely generated one-relator parafree groups (introduced by Baumslag): residually nilpotent groups and its quotients by the terms of its lower central series are the same as those of a free group.

(3) Right-angled Artin groups A_G, where

G is a graph with vertices $V(G) = \{1, 2, \ldots, n\}$ for $n \geq 2$ and edges $E(G)$,

$$A_G = \langle x_1, \ldots, x_n \mid [x_i, x_j], (i, j) \in E(G) \rangle$$

- (Harizanov, Knight, McCoy, Puzarenko, Solomon and Wallbaum)

  Let $F_\infty = \langle x_0, x_1, \ldots \mid \ \rangle$ be a free group of rank $\aleph_0$.
  There is a computable copy F of F_∞
  with no computable left order (hence no computable bi-order).

- *Corollary*
  The space $LO(F_\infty)$ and the space $BiO(F_\infty)$ are homeomorphic
  to the Cantor set.

- For every left-orderable computable group G, there is a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection from $LO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- An isolated path in a computable binary tree must be computable.

- In a computable binary tree with infinite paths and no computable ones, the space of paths is homeomorphic to the Cantor set.

- Topology of the space of orders does not change under isomorphisms, so it is the same for the whole isomorphism class.

Complexity of a Basis of F_∞

- (Carson, Harizanov, Knight, Lange, McCoy, Morozov, Quinn, Safranski and Wallbaum) Let G be a computable isomorphic copy of F_∞.
  Then G has a Π^0_2 basis.

- *Proof sketch*

  There is a computable sequence of computable Π_2 formulas $\gamma_k(x_1, \ldots, x_k)$ so that a k-tuple $(a_1, \ldots, a_k) \in G$ is part of a basis iff $G \models \gamma_k(a_1, \ldots, a_k)$. This can be used to obtain a Δ^0_3 basis $B = \{b_0, b_1, \ldots\}$. Now we use B to produce a Π^0_2 basis U. We give a Δ^0_2 enumeration of the complement $\overline{U}$. We use the fact that given a basis $\{x, y\}$ for a free group of rank 2, we can apply Nielsen transformations to obtain infinitely many further bases, all disjoint. Starting with $\{x, y\}$, we get $\{xy, y\}$, and then $\{xy, xy^2\}$, disjoint from $\{x, y\}$.

- A *Nielsen transformation* on a tuple $(x_1, ..., x_n)$ is the result of finitely many applications of the following steps:

    1. replace x_i by x_i^{-1};

    2. replace x_i, x_j by $x_i x_j, x_j$;

    3. for some i such that $a_i = e$ delete a_i.

- If U is carried into V by Nielsen transformation, then $Gr(U) = Gr(V)$.

- *Example*. Suppose that $\{x, y\}$ is a basis for F_2. Then the following are also bases:

  $\{xy, y\}$, $\{xy, xy^2\}$, $\{xyxy^2, xy^2\}$, etc.

- Let G be a computable isomorphic copy of F_∞.
  If G has a $\mathbf{\Sigma}^0_2$ basis, then G has a $\mathbf{\Delta}^0_2$ basis.

- (McCoy and Wallbaum) There is a computable isomorphic copy of F_∞ with no $\mathbf{\Sigma}^0_2$ basis.

- *Proof* uses the following result:

  (Bestvina and Feighn) Let G be a free group generated by a, b, c. The word $a^2b^2c^3$ is not primitive. However, this word satisfies all $\mathbf{\Pi}_1$ formulas true of a basis element.

- Let $w_1(\overline{x}), \ldots, w_k(\overline{x})$ be a k-tuple of words on an n-tuple of variables $\overline{x}$, where $k \leq n$. The tuple of words is called *primitive* if whenever the tuple $\overline{x}$ forms a basis for F_n, the tuple $w_1(\overline{x}), \ldots, w_k(\overline{x})$ forms part of the basis.

Describing Free Groups

- (Sela) Let $m, n \geq 2$. Then F_m and F_n are elementarily equivalent.

- Describing different free groups is possible using $L_{\omega_1\omega}$-sentences by Scott Isomorphism Theorem.

- We can describe free groups using computable infinitary sentences. We use tools of computability theory to obtain optimal description.

- For a computable structure A, a *computable index* is a number e such that φ_e is the characteristic function of the atomic diagram of A.

  $(\varphi_e)_{e\in\omega}$ partial computable functions

- The *index set* for A, denoted by $I(A)$, is the set of all computable indices for isomorphic copies of A.

  The *index set* of a class of structures K (closed under isomorphism), denoted by $I(K)$, is the set of all computable indices for members of K.

- Let Γ be a complexity class.

  We define $I(A)$ to be Γ *within* K if $I(A) = C \cap I(K)$ for some $C \in \Gamma$.

  We say that $I(A)$ is Γ-*hard within* K if for every $S \in \Gamma$, there is a computable function $f : \omega \longrightarrow I(K)$ such that

$$n \in S \Leftrightarrow f(n) \in I(A)$$

- Furthermore, $I(A)$ is m-*complete* Γ *within* K if $I(A)$ is Γ within K, and $I(A)$ is Γ-hard within K.

(Carson et al.)

- *Within the class of free groups:*

  (1) *The index set* $I(F_1)$ *is* m*-complete* Π^0_1

  (2) *The index set* $I(F_2)$ *is* m*-complete* Π^0_2

  (3) *For* $n > 2$, *the index set* $I(F_n)$ *is* m*-complete* d-Σ^0_2

  (4) *The index set* $I(F_\infty)$ *is* m*-complete* Π^0_3

- *Within the class of all groups:*

  (5) *For* $n \geq 1$, *the index set* $I(F_n)$ *is* m*-complete* d-Σ^0_2.

  (6) (McCoy and Wallbaum) *The index set* $I(F_\infty)$ *is* m*-complete* Π^0_4

Examples

- Describing F_1 within FG (the class of free groups):

  There is a (finitary) Π_1 sentence saying that the group is abelian.

  For hardness can show that for any Π^0_1 set S, there is a uniformly computable sequence of structures $(C_n)_{n\in\omega}$ such that

$$C_n = \begin{cases} F_1 & \text{if } n \in S \\ F_2 & \text{if } n \notin S \end{cases}$$

- For $n \geq 1$, there is a computable Π_2 sentence σ_n saying that for every $(n+1)$-tuple of elements there is an n-tuple that generates it.

- Describing F_2 within FG:

  We describe F_2 within FG by the conjunction of σ_2 and a finitary $\boldsymbol{\Sigma}_1$ sentence saying that the group is not abelian.

  For hardness can show that for any Π^0_2 set S, there is a uniformly computable sequence of structures $(C_n)_{n\in\omega}$ such that

$$C_n = \begin{cases} F_2 & \text{if } n \in S \\ F_3 & \text{if } n \notin S \end{cases}$$

- Describing F_n within FG for $n > 2$:

  We describe F_n within FG by the conjunction of σ_n and the negation of σ_{n-1}.

  For hardness can show that for any $\mathbf{\Sigma}^0_2$ sets S_1 and S_2, there is a uniformly computable sequence of structures $(C_n)_{n\in\omega}$ such that

$$C_n = \begin{cases} F_{n-1} & \text{if } n \notin S_1 \\ F_n & \text{if } n \in S_1 \wedge n \notin S_2 \\ F_{n+1} & \text{if } n \in S_1 \cap S_2 \end{cases}$$

- Hardness of F_∞ within FG uses the set Cof $= \{e : dom(\varphi_e) \text{ is cofinite}\}$ m-complete $\mathbf{\Sigma}^0_3$ set.

  Hardness of F_∞ within the class of all groups uses previously mentioned Bestvina-Feighn's result.

THANK YOU!