Model theory of operator algebras

Bradd Hart
McMaster University

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Outline

- Background on operator algebras
- Some basics of the model theory of C^*-algebras
- Some basics of the model theory of II_1 factors
- Results and open questions about II_1 factors
- Results and open questions about C^*-algebras

Linear operators

- Fix a Hilbert space H and consider a linear operator A on H. The operator norm of A is defined by

$$\|A\| := sup\{\frac{\|Ax\|}{\|x\|} : x \in H, x \neq 0\}$$

  if this is defined and if it is, we call A bounded.
- We write $B(H)$ for the set of all bounded operators on H.
- $B(H)$ carries a natural complex vector space structure; it also has a multiplication given by composition. There is an adjoint operation defined via the inner product on H: for $A \in B(H)$, A^* satisfies, for all $x, y \in H$,

$$\langle Ax, y\rangle = \langle x, A^* y\rangle.$$

- The operator norm puts a normed linear structure on $B(H)$ and the norm satisfies the C*-identity $\|A^* A\| = \|A\|^2$ for all $A \in B(H)$.

C^*-algebras

Definition

- A *concrete* C^*-algebra is a norm closed *-subalgebra of $B(H)$ for some Hilbert space H.
- An *abstract* C^*-algebra is Banach *-algebra which satisfies the C^*-identity.

Theorem (Gelfand-Naimark-Segal)

Every abstract C^*-algebra is isomorphic to a concrete C^*-algebra.

Example

- For any Hilbert space H, $B(H)$ is a concrete C*-algebra. In particular, $M_n(\mathbb{C})$, $n \times n$ complex matrices, is a C^*-algebra for all n.
- $C(X)$, all complex-valued continuous functions on a compact, Hausdorff space X is an abelian (abstract) C^*-algebra. The norm is the sup-norm. By a result of Gelfand and Naimark, these are all the unital abelian C^*-algebras.

C^*-algebraic ultraproducts

Suppose A_i are C^*-algebras for all $i \in I$ and that $\mathcal{U}$ is an ultrafilter on I. Consider the bounded product

$$\prod^b A_i := \{\bar{a} \in \prod A_i : \lim_{i \to \mathcal{U}} \|a_i\| < \infty\}$$

and the two-sided ideal $c_{\mathcal{U}}$

$$\{\bar{a} \in \prod^b A_i : \lim_{i \to \mathcal{U}} \|a_i\| = 0\}.$$

The ultraproduct, $\prod_{\mathcal{U}} A_i$ is defined as $\prod^b A_i / c_{\mathcal{U}}$.

C*-algebras as metric structures

- C*-algebras are treated as metric structures using bounded balls; there are sorts for each ball of radius $n \in \mathbb{N}$.
- There are inclusion maps between the balls. Additionally there are functions for the restriction of all the operations to the balls. This involves the addition, multiplication, scalar multiplication and the adjoint.
- The metric is given via the operator norm as $\|x - y\|$ on each ball.
- It is routine to check that all of these functions are uniformly continuous (the only issue is multiplication and this holds because we have restricted the norm).
- The sorts are complete since C*-algebras are complete.
- (FHS) The class of C*-algebras forms an elementary class in this language.

A second topology

- The weak operator topology on $B(H)$ is induced by the family of semi-norms given by, for every $\zeta, \eta \in H$,

$$A \mapsto |\langle A\zeta, \eta \rangle|.$$

- $M \subseteq B(H)$ is a von Neumann algebra if it is a unital *-algebra closed in the weak operator topology.
- Equivalently, any unital *-algebra $M \subseteq B(H)$ which satisfies $M'' = M$ is a von Neumann algebra where

$$M' = \{A \in B(H) : [A, B] = 0 \text{ for all } B \in M\}.$$

- A linear functional τ on a C*-algebra M is a (finite, normalized) **trace** if it is positive ($\tau(a^*a) \geq 0$ for all $a \in M$), $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$ and $\tau(1) = 1$. We say it is faithful if $\tau(a^*a) = 0$ implies $a = 0$.

Tracial von Neumann algebras

A tracial von Neumann algebra M is a von Neumann algebra with a faithful trace τ. τ induces a norm on M

$$\|a\|_2 = \sqrt{\tau(a^*a)}.$$

Example

- $M_n(\mathbb{C})$ with the normalized trace; $B(H)$ for infinite-dimensional H is not.
- Inductive limits of tracial von Neumann algebras are tracial von Neumann algebras. In particular, $\mathcal{R}$, **the** inductive limit of the $M_n(\mathbb{C})$'s is a tracial von Neumann algebra called the hyperfinite II_1 factor.
- $L(F_n)$ - suppose H has an orthonormal generating set ζ_h for $h \in F_n$. Let u_g for $g \in F_n$ be the operator determined by

  $$u_g(\zeta_h) = \zeta_{gh}.$$

  $L(F_n)$ is the von Neumann algebra generated by the u_g's. It is tracial: for $a \in L(F_n)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$.

Tracial vNas as metric structures

- As with C*-algebras, for a tracial von Neumann algebra, we introduce sorts for the balls of operator norm n for each $n \in \mathbb{N}$.
- The basic functions are again considered as partitioned across the sorts.
- The metric on each ball is induced by the 2-norm.
- (FHS) The class of tracial von Neumann algebras forms an elementary class
- Tracial ultraproducts of von Neumann algebras, introduced by McDuff, are also equivalent to the ultraproduct in the metric structure sense for tracial von Neumann algebras.
- A von Neumann algebra whose centre is $\mathbb{C}$ is called a factor. A tracial factor is type II_1 if it contains a projection with irrational trace. (FHS) The class of II_1 factors is elementary.
- $\mathcal{R}$, $\mathcal{R}^{\mathcal{U}}$, $\prod_U M_n(\mathbb{C})$ and $L(F_n)$ are all II_1 factors.

Property Γ

- Consider M any II_1 factor and the partial type

$$p(x) = \{[x, m] = 0 : m \in M\}.$$

- (MvN) M has property Γ if p is not algebraic in the theory of M. Property Γ is elementary by its definition.
- $\prod_{\mathcal{U}} M_n(\mathbb{C})$ does not have property Γ; neither does $L(F_n)$.
- Consider $M \prec M^{\mathcal{U}}$ and all realizations of p in $M^{\mathcal{U}}$ - it is $M' \cap M^{\mathcal{U}}$, the relative commutant or the central sequence algebra. It is also a von Neumann algebra.
- There are three cases (McDuff):
  - M does not have property Γ,
  - M has property Γ and the relative commutant is abelian (and does not depend on U), or
  - M has a non-abelian relative commutant (it is type II_1).
- McDuff asked if in the third case, the isomorphism type depends on $\mathcal{U}$. FHS answered yes because the theory of II_1 factors is unstable!

The theory of $\mathcal{R}$

- $\mathcal{R}$ is the atomic model of its theory; any embedding of it into any other model of its theory is automatically elementary.
- *Th*($\mathcal{R}$) is not model complete; in particular, it does not have quantifier elimination (FGHS; GHS).
- A question logicians must ask: is the theory of $\mathcal{R}$ decidable?
- What does this mean for a continuous theory? Is there an algorithm such that given a sentence φ and $\epsilon > 0$, we can compute $\varphi^{\mathcal{R}}$ to within ϵ.
- By (BYP), the answer is yes if there is a recursive axiomatization of *Th*($\mathcal{R}$).
- Do we know such an axiomatization? No!
- We do have a recursive axiomatization of all tracial von Neumann algebras - this is a universal class so what do we know about $Th_{\forall}(\mathcal{R})$? Is it decidable?

A little background

- If A is any separable II_1 tracial von Neumann algebra then $\mathcal{R} \hookrightarrow A$;
- If $A \equiv_\forall \mathcal{R}$ then $A \hookrightarrow \mathcal{R}^\mathcal{U}$.
- Equivalently, if $A \hookrightarrow \mathcal{R}^\mathcal{U}$ then $Th_\forall(A) = Th_\forall(\mathcal{R})$.
- So if all separable II_1 tracial von Neumann algebras embed into $\mathcal{R}^\mathcal{U}$ then $Th_\forall(\mathcal{R})$ is decidable.

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- So if all separable II_1 tracial von Neumann algebras embed into $\mathcal{R}^\mathcal{U}$ then $Th_\forall(\mathcal{R})$ is decidable.
- Problem: the assumption is the Connes Embedding Problem!
- In fact, CEP is equivalent to the decidability of the universal theories of all tracial von Neumann algebras. (GH)
- To me, this says that this problem is very hard or that $Th(\mathcal{R})$ is undecidable (or both).

Theories of II_1 factors

- (BCI; GH2; GHT) There are continuum many theories of II_1 factors. In fact, McDuff's original examples of continuum many non-isomorphic II_1 factors are not elementarily equivalent.
- The free group factor problem asks if $L(F_m)$ and $L(F_n)$ are not isomorphic for $m \neq n$. A model theoretic version of this question: are $L(F_m)$ and $L(F_n)$ elementarily equivalent?
- We do know that $L(F_\infty)$ and $\prod_{\mathcal{U}} L(F_n)$ have the same $\forall\exists$-theory for non-principal $\mathcal{U}$.
- Related questions: for non-principal $\mathcal{U}$, is the theory of $\prod_{\mathcal{U}} M_n(\mathbb{C})$ independent of $\mathcal{U}$? How are the theory of ultraproducts of matrix algebras related to the theories of free group factors?

Nuclear algebras

- A linear map $\varphi : A \to B$ is positive if $\varphi(a^* a) \geq 0$ for all $a \in A$ (positive elements go to positive elements).
- φ is completely positive if for all n, $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive.
- φ is contractive if $\|\varphi\| \leq 1$; *-homomorphisms are cpc maps.

Definition

A C*-algebra A is nuclear if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an n and cpc maps

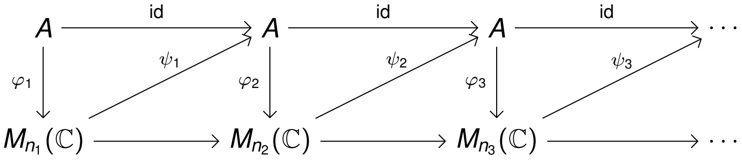
$$\varphi : A \to M_n(\mathbb{C}) \text{ and } \psi : M_n(\mathbb{C}) \to A$$

such that

$$\|\bar{a} - \psi\varphi(\bar{a}))\| < \epsilon.$$

- Examples: Abelian C*-algebras, $M_n(\mathbb{C})$
- Inductive limits of nuclear algebras; nuclear algebras are closed under $\otimes$ and direct sum.

A helpful picture



Model theoretic characterization of nuclear algebras

- The general classification problem is to give a complete (usable) set of invariants for all (unital), separable, simple nuclear algebras.
- Consider, for $k, n \in \mathbb{N}$, the predicate defined on A_1^k by

$$R_n^k(\bar{a}) = \inf_{\varphi,\psi} \|\bar{a} - \psi(\varphi(\bar{a}))\|$$

  where $\varphi : A \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to A$ range over cpc maps.
- (FHLRTVW) This predicate is a definable predicate in the language of C*-algebras by the Beth definability theorem.
- It follows then that a C*-algebra is nuclear if it satisfies, for all k,

$$\inf_{\bar{x}} \inf_{n} R_n^k(\bar{x}).$$

Strongly self-absorbing algebras

- D is an ssa algebra if $D \cong D \otimes D$ and this isomorphism is approximately unitarily equivalent to $id_D \otimes 1$.
- Conjecturally, all ssa algebras are known: $\mathcal{Z}$, $M_{p^\infty}(\mathbb{C})$, $\mathcal{O}_\infty$ and $\mathcal{O}_2$ together with their (possibly infinite) tensor products.
- The model theory of an ssa algebra is very nice: If D is ssa then
  - it is the prime model of its theory, and
  - any embedding of D into a model of the theory of D is automatically elementary.
- (FHRT) D is ssa iff
  - if whenever $\mathcal{U}$ is an non-principal ultrafilter on $\mathbb{N}$ and $\sigma : D \to D^{\mathcal{U}}$ is elementary then σ is unitarily equivalent to the diagonal map, and
  - $D \equiv D \otimes D$.

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