The Halpern-Läuchli Theorem at a Measurable Cardinal

Dan Hathaway, joint with Natasha Dobrinen

University of Denver

Daniel.Hathaway@du.edu

March 19, 2017

Basic Definitions

Definition

Let κ be a regular cardinal. A tree $T \subseteq {}^{<\kappa}\kappa$ is **regular** iff it is

1) perfect,
2) suitable (every maximal branch has length κ), and
3) a κ-tree (every level $T(\alpha) := T \cap {}^{\alpha}\kappa$ of T has size $< \kappa$).

Note: If κ is not strongly inaccessible, there are no regular trees.

Definition

Given sets $T_0, ..., T_{d-1} \subseteq {}^{<\kappa}\kappa$, $T_0 \otimes ... \otimes T_{d-1}$ is the set of d-tuples $\langle t_0, ..., t_{d-1} \rangle$ such that each $t_i \in T_i$ and the t_i's are all on the same level.

Definition

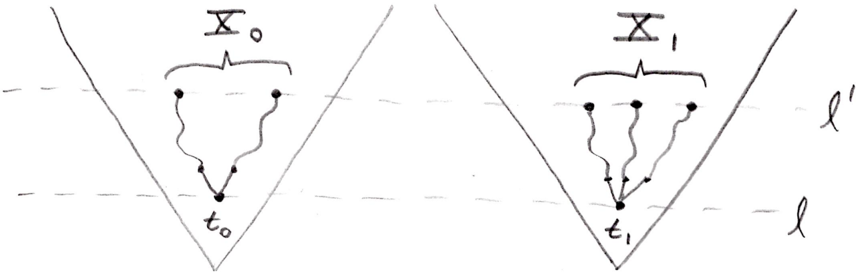
Given sets $A, X \subseteq {}^{<\kappa}\kappa$, we say that X **dominates** A iff each $a \in A$ is extended by some $x \in X$.

Definitions: SDHL

SDHL stands for "Somewhere Dense Halpern-Läuchli".

Definition

Given a cardinal $\sigma > 0$, SDHL(d, σ, κ) is the statement that given any regular trees $T_0, \ldots, T_{d-1} \subseteq {}^{<\kappa}\kappa$ and any level coloring $c : T_0 \otimes \ldots \otimes T_{d-1} \to \sigma$, there are levels $l < l' < \kappa$, a sequence of nodes $\langle t_i \in T_i(l) : i < d \rangle$, and a sequence of sets $\langle X_i \subseteq T_i(l') : i < d \rangle$ such that each X_i dominates $\mathrm{Succ}_{T_i}(t_i)$ and c is constant on $X_0 \otimes \ldots \otimes X_{d-1}$.



Definitions: SSHL

SSHL stands for "Strong Subtree Halpern-Läuchli", and it implies SDHL.

Definition

Given regular trees $S \subseteq T \subseteq {}^{<\kappa}\kappa$, we say that S is a **strong** subtree of T as witnessed by $A \in [\kappa]^\kappa$ iff for each $l \in \kappa$ and $t \in S(l)$,

1) If $l \notin A$, then $|\text{Succ}_S(t)| = 1$;
2) If $l \in A$, then $\text{Succ}_S(t) = \text{Succ}_T(t)$.

Definition

Given a cardinal $\sigma > 0$, SSHL(d, σ, κ) is the statement that given any regular trees $T_0, \ldots, T_{d-1} \subseteq {}^{<\kappa}\kappa$ and any level coloring $c : T_0 \otimes \ldots \otimes T_{d-1} \to \sigma$, there are strong trees $S_0 \subseteq T_0$, ..., $S_{d-1} \subseteq T_{d-1}$ all witnessed by the same set of levels $A \in [\kappa]^\kappa$ and $(\forall l \in A)$ c is constant on $S_0(l) \otimes \ldots \otimes S_{d-1}(l)$.

For the rest of this presentation, assume $0 < d < \omega$ and $0 < \sigma < \kappa$.

Complexity and reflection at a measurable

SDHL(d, σ, κ) is a $\mathbf{\Pi}_1$ statement about $V_{\kappa+1}$. Let M be a model of ZF such that $V_\kappa \subseteq M$. If SDHL(d, σ, κ) is true in V, then it is true in M. If $V_{\kappa+1} \subseteq M$, then the other direction holds.

SSHL(d, σ, κ) is a $\mathbf{\Pi}_2$ statement about $V_{\kappa+1}$.

Proposition (D., H.)

Let κ be a measurable cardinal with a normal measure $\mathcal{U}$. Fix d and $\sigma < \kappa$. Then SDHL(d, σ, κ) iff

$$\{\alpha < \kappa : \mathrm{SDHL}(d, \sigma, \alpha)\} \in \mathcal{U}.$$

The same is true for SSHL(d, σ, κ) in place of SDHL.

Proof: Let $j : V \to M$ be the ultrapower embedding. Because $V_{\kappa+1} \subseteq M$, SDHL$(d, \sigma, \kappa) \Leftrightarrow$ SDHL$(d, \sigma, \kappa)^M$. By Łos's Theorem, SDHL$(d, \sigma, \kappa)^M \Leftrightarrow \{\alpha < \kappa : \mathrm{SDHL}(d, \sigma, \alpha)\} \in \mathcal{U}$. The same argument works for SSHL(d, σ, κ) in place of SDHL.

Upwards stationary reflection for SDHL

We do NOT know if the following holds for SSHL instead.

Proposition (D., H.)

Assume that

$$S := \{\alpha < \kappa : \mathrm{SDHL}(d, \sigma, \alpha)\}$$

is stationary. Then $\mathrm{SDHL}(d, \sigma, \kappa)$ holds.

Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < d \rangle$ be a sequence of regular trees and let $c : \bigotimes_{i<d} T_i \to \sigma$ be a coloring. If we can find an $\alpha < \kappa$ such that each $T_i \cap {}^{<\alpha}\kappa$ is an α-tree and $\mathrm{SDHL}(d, \sigma, \alpha)$ holds, then we will be done. An elementary argument shows that for each $i < d$, there is a club $C_i \subseteq \kappa$ such that $(\forall \alpha \in C_i)$ $T_i \cap {}^{<\alpha}\kappa$ is an α-tree. The set $\bigcap_{i<d} C_i$ is a club, so it must intersect S. An $\alpha < \kappa$ in the intersection is as desired.

Corollary

If $\mathrm{SDHL}(d, \sigma, \alpha)$ holds for a stationary set of $\alpha < \kappa$, then $\mathrm{SDHL}(d, \sigma, \kappa)$ holds in V and in any $<\kappa$-closed forcing extension.

SDHL implies SSHL at a weakly compact

When κ is weakly compact or ω, then SDHL and SSHL are equivalent:

Theorem (see [7])

Let κ be weakly compact (or ω) and fix d and σ. Then SDHL(d, σ, κ) implies SSHL(d, σ, κ) .

A noteworthy fact is that for any κ and any *finite* $\sigma < \omega$, SDHL$(1, \sigma, \kappa)$ holds. By the theorem above, when κ is weakly compact or ω, then SSHL$(1, \sigma, \kappa)$ holds.

Question: Can SDHL(d, σ, κ) hold but SSHL(d, σ, κ) fail? In fact, it is unknown whether it is consistent that SSHL(d, σ, κ) (and therefore SDHL(d, σ, κ)) can fail anywhere!

Proving SSHL

SSHL(d, σ, ω) can be proved by induction on $d < \omega$ (see [7]). The successor step involves a fusion argument. This cannot be generalized to the $\kappa > \omega$ case because the intersection of a decreasing sequence of regular trees may not be regular.

There is another proof of SSHL(d, σ, ω) (see [3]) which adds many Cohen reals by forcing, and uses an ultrafilter in the extension to make selections. This generalizes to the $\kappa > \omega$ case if we assume that κ is measurable in the extension:

Theorem (see [1])

Let $\lambda > \kappa$ satisfy $\lambda \to (\kappa)^d_\kappa$. Assume κ is measurable in the forcing extension where we add λ many Cohen subsets of κ. Then SSHL(d, σ, κ) holds (in the ground model).

In [6], there is a theorem with a similar hypothesis and the conclusion implies SSHL$(1, \sigma, \kappa)$ for *all* $\sigma < \kappa$, not just finite $\sigma < \omega$ which we mentioned was true before.

The cardinal κ is α-strong iff there is an elementry embedding $j : V \to M$ such that crit$(j) = \kappa$ and $V_{\kappa+\alpha} \subseteq M$. By a (slight modification of a) theorem of Woodin (see [4] for a proof), if GCH holds and κ is $(\kappa + d)$-strong, then there is a forcing extension in which κ is measurable and remains measurable after adding $\lambda = \kappa^{+d}$ Cohen reals. This gives us the following:

Corollary

Assume there is a model in which GCH holds and there is a cardinal κ which is $(\kappa + d)$-strong. Then there is a forcing extension in which κ is measurable and SSHL(d, σ, κ) holds.

Question: is the existence of a $(\kappa + d)$-strong cardinal equiconsistent with there existing a measurable κ such that $(\forall \sigma < \kappa)$ SSHL(d, σ, κ) ?

Preservation by small forcings

The following works for SSHL in place of SDHL.

Theorem (D., H.)

Let $\mathbb{P}$ be a forcing of size $< \kappa$. Then $\mathrm{SDHL}(d, \sigma \cdot |\mathbb{P}|, \kappa)$ implies $1 \Vdash_{\mathbb{P}} \mathrm{SDHL}(d, \sigma, \kappa)$.

Sketch ($d = 2$ case): Given a name $\dot{T}$ for a regular tree, let $\mathrm{Der}(\dot{T})$ be the set of all equivalence classes of pairs $(\dot{\tau}, \alpha)$ such that

$$1 \Vdash_{\mathbb{P}} (\dot{\tau} \in \dot{T} \text{ and } \mathrm{Length}(\dot{\tau}) = \check{\alpha}),$$

where $(\dot{\tau}_1, \alpha_1) \cong (\dot{\tau}_2, \alpha_2)$ iff $1 \Vdash_{\mathbb{P}} (\dot{\tau}_1 = \dot{\tau}_2)$. Order $\mathrm{Der}(\dot{T})$ by $[(\dot{\tau}_1, \alpha_1)] \le [(\dot{\tau}_2, \alpha_2)]$ iff $1 \Vdash_{\mathbb{P}} \dot{\tau}_1 \sqsubseteq \dot{\tau}_2$. Fact: $\mathrm{Der}(\dot{T})$ is a regular tree. Given names $\dot{T}_1$, $\dot{T}_2$ for regular trees and a name $\dot{c}$ such that $1 \Vdash_{\mathbb{P}} [\dot{c} : \dot{T}_1 \otimes \dot{T}_2 \to \check{\sigma}]$, let $c : \mathrm{Der}(\dot{T}_1) \otimes \mathrm{Der}(\dot{T}_2) \to \mathbb{P} \times \sigma$ be any coloring such that for each $r = \langle (\dot{\tau}_1, \alpha), (\dot{\tau}_2, \alpha) \rangle$,

$$\mathrm{First}(c(r)) \Vdash_{\mathbb{P}} \dot{c}(\dot{\tau}_1, \dot{\tau}_2) = \mathrm{Second}(c(r)).$$

SDHL at a not weakly compact cardinal

Corollary

If GCH holds and κ is $(\kappa + d)$-strong, then there is a forcing extension in which $(\forall \sigma < \kappa)\,\mathrm{SDHL}(d, \sigma, \kappa)$ holds, but κ is not weakly compact.

Proof: First force over V to get a model $V[G_1]$ in which SDHL holds at κ, which is also measurable. By a theorem of Hampkins, any non-trivial forcing of size $< \kappa$ followed by a non-trivial $<\kappa$-closed forcing will make κ NOT weakly compact. Perform any non-trivial forcing of size $< \kappa$ over $V[G_1]$ to get $V[G_1][G_2]$. This will preserve SDHL at κ by the previous theorem. Since κ is measurable in $V[G_1][G_2]$, SDHL holds on a stationary (in fact, measure one) subset of κ. Now perform any non-trivial $<\kappa$-closed forcing over $V[G_1][G_2]$ to get $V[G_1][G_2][G_3]$. Stationary subsets of κ are preserved, so inside $V[G_1][G_2][G_3]$, SDHL holds on a stationary subset of κ. Thus, SDHL holds at κ in this model.

Question: does SDHL or SSHL have any large cardinal strength (beyond that of a strongly inaccessible, which is needed for the definition)?

References

- N. Dobrinen and D. Hathaway. *The Halpern-Läuchli Theorem at a Measurable Cardinal*. Submitted (see arXiv).
- M. Džamonja, J. Larson, and W. J. Mitchell. *A partition theorem for a large dense linear order*. Israel Journal of Mathematics 171 (2009).
- I. Farah and S. Todorcevic. *Some Applications of the Method of Forcing* (1995).
- S. Friedman and K. Thompson. *Perfect trees and elementary embeddings*. The Journal of Symbolic Logic 73 (2008).
- J. D. Halpern and H. Lauchli. *A partition theorem*. Transactions of the American Mathematical Society 124 (1966).
- S. Shelah. *Strong partition relations below the power set: consistency - was Sierpinski right? II*. Sets, Graphs and Numbers 60 (1991).
- S. Todorcevic. *Introduction to Ramsey Spaces* (2010).

Thank You!