Metastability and Model Theory

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The talk will be divided into four sections.

Section 1  Historical background.
Section 2  Uniform metastability
Section 3  Metastable convergence theorems.
Section 4  Metastability and compactness.
Ongoing research in collaboration with Eduardo Duenez (supported by NSF grant DMS-1500615)

The last result is part of ongoing research with Eduardo Dueñez and Xavier Caicedo (partially supported by NSF grant DMS-11445110).
A measure preserving system is a structure of the form 
\((X, \mathcal{X}, \mu, T)\), where \((X, \mathcal{X}, \mu)\) is a probability space and \(T : (X, \mathcal{X}, \mu) \to (X, \mathcal{X}, \mu)\) is a probability space isomorphism. In particular,

- \(T\) is invertible,
- \(T\) and \(T^{-1}\) are measurable,
- \(\mu(T^nE) = \mu(T(E))\) for every \(E \in \mathcal{X}\) and every integer \(n\).

We are interested in recurrence properties of sets \(E \in \mathcal{X}\), or functions \(f \in L^p(X, \mathcal{X}, \mu)\).
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Poincaré recurrence theorem

Let \((X, \mathcal{X}, \mu, T)\) be a measure-preserving system, and let \(E \in \mathcal{X}\) be such that \(\mu(E) > 0\). Then,

\[
\limsup_{n \to \infty} \mu(E \cap T^n E) \geq \mu(E)^2.
\]

In particular, \(\mu(E \cap T^n E) > 0\), for infinitely many \(n\).
Von Neumann ergodic theorem (1932)

Let \( U : H \rightarrow H \) be a unitary operator on a separable Hilbert space \( H \). Then the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n v
\]

exists for every \( v \in H \). Moreover, the limit equals \( \pi(v) \), where \( \pi \) is the orthogonal projection from \( H \) onto the closed subspace \( \{ v \in H : Uv = v \} \) consisting of all \( U \)-invariant vectors.
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$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n v$$

exists for every $v \in H$. Moreover, the limit equals $\pi(v)$, where $\pi$ is the orthogonal projection from $H$ onto the closed subspace $\{v \in H : Uv = v\}$ consisting of all $U$-invariant vectors.
Corollary (Mean ergodic theorem)

Let \((X, \mathcal{X}, \mu, T)\) be a measure-preserving system. Then the limit of averages

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f
\]

exists for every \(f \in L^2(X, \mathcal{X}, \mu)\)
Furstenberg’s proof of Szemerédi’s theorem (1977) and its subsequent refinement by Furstenberg and Katznelson (1978) suggested at least three possible directions of generalization:

1. Replacing $n \mapsto T^n$ by more general group actions, (i.e., $\mathbb{Z}$ by other groups),

2. Considering polynomial, rather than linear actions,

3. Establishing uniform bounds for convergence.

For this talk we will restrict our attention to (3). Let us first start with a well-known example:
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Furstenberg multiple recurrence theorem

Theorem (Furstenberg, 1977)
Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then for every set $E$ of positive measure and every positive integer $k$ there exists $n > 0$ such that

$$E \cap T^{-n}E \cap \ldots \cap T^{-(k-1)n}E \neq \emptyset.$$
Uniform Furstenberg multiple recurrence theorem

Theorem (Bergelson, Host, McCutcheon, Parreau, 2000)

For every positive integer $k$ and every $\delta > 0$ there exists $\epsilon(k, \delta) > 0$ with the following property: For every measure-preserving system $(X, \mathcal{X}, \mu, T)$ and every measurable set $E$ with $\mu(E) \geq \delta$, 

$$\frac{1}{N} \sum_{k=0}^{N-1} \mu(E \cap T^n E \cap \ldots \cap T^{(k-1)n} E) \geq \epsilon(k, \delta),$$

for all $N \geq 1$. 

The norm convergence problem for several commuting transformations

**Theorem (Tao, 2007)**

If \((X, \mathcal{X}, \mu)\) is a probability space and \(T_1, \ldots, T_k : X \to X\) are commuting measure-preserving transformations, then for any bounded measurable functions \(f_1, \ldots, f_k : X \to \mathbb{R}\), the multiple averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} T_1^n f_1 \cdots T_k^n f_k
\]

converge in the \(L^2(X)\) norm topology (and hence in probability) as \(N \to \infty\).
The case $k = 1$ is Von Neumann’s mean ergodic theorem
The case $k = 2$: Conze and Lesigne (1983)
The case for higher $l$ was established by Frantzikinakis and Kra (2005) under additional hypothesis for the operators $T_i$.
The case $T_i = T^i$: Host-Kra (2005), Ziegler (2007)
Remark: Tao’s argument does not establish a formula for the limit of the sequence of averages. He rather proves that the sequence converges indirectly, by showing that is Cauchy in $L^2(X)$.

For this, he introduces the concept of metastability of sequences and metastable convergence. A crucial component of his proof is his Metastable Dominated Convergence Theorem.

The concept of metastability has been studied from the perspective of computable analysis by Avigad-Dean-Rute, Avigad-Towsner, Kohlenbach, Kohlenbach-Leustea, Kohlenbach-Safarik, Körnlein-Kohlenbach, and Schade-Kohlenbach.
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Walsh’s convergence theorem

Theorem (Walsh, 2012)
Let \((X, \mu)\) be a measure space with a measure-preserving action of a nilpotent group \(G\). Let \(g_1, \ldots, g_k : \mathbb{Z} \to G\) be polynomial sequences in \(G\) (i.e. each \(g_i\) is of the form \(g_i(n) = a_{i,1}^{p_{i,1}(n)} \cdots a_{i,j}^{p_{i,j}(n)}\) for some \(a_{i,1}, \ldots, a_{i,j} \in G\) and polynomials \(p_{i,1}, \ldots, p_{i,j} : \mathbb{Z} \to \mathbb{Z}\). Then for any \(f_1, \ldots, f_k \in L^\infty(X, \mu)\), the averages
\[
\frac{1}{N} \sum_{n=1}^{N-1} (g_1(n)f_1) \cdots (g_k(n)f_k)
\]
converge in \(L^2(X, \mu)\) norm as \(N \to \infty\), where
\(g(n)f(x) := f(g(n)^{-1}x)\).
Remarks:

- Walsh’s argument, like Tao’s, relies heavily on *metastability*.
- Nilpotence plays a crucial role in Walsh’s proof. A key part of his argument uses Leibman’s theory of polynomials maps of groups (1998–2002), which relies heavily on nilpotence. Nilpotence is widely regarded as the *non plus ultra* condition ensuring $L^2$-convergence of multiple ergodic averages.
Definitions: Samplings and metastability rates

Definition
A *sampling* of the totally ordered set \((\mathbb{N}, <)\) is a function

\[
\eta : \mathbb{N} \to \mathbb{N}
\]

such that \(\eta(n) \geq n\) for all \(n \in \mathbb{N}\). The set of all samplings of \(\mathbb{N}\) will be denoted \(\text{Smpl}(\mathbb{N})\). To each sampling \(\eta\) there corresponds the collection of intervals \([n, \eta(n)] \subset \mathbb{N}\), one for each \(n \in \mathbb{N}\).

Definition
A *rate of metastability* is a family of natural numbers, one for each \(\epsilon > 0\) and \(\eta \in \text{Smpl}(\mathbb{N})\).

\[
E_\bullet = (E_{\epsilon, \eta}) \subset \mathbb{N}
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Metastability for sequences with a given rate

Definition ([Tao])

For each sampling $\eta$ and $\epsilon > 0$ let $E_{\epsilon, \eta} \in \mathbb{N}$ be given.

- A sequence $(a_n)_{n \in \mathbb{N}}$ in a metric space $(X, d)$ is $[\epsilon, \eta]$-metastable (with bound $E_{\epsilon, \eta}$) if there exists $n$ (no larger than $E_{\epsilon, \eta}$) such that

\[ d(a_m, a_{m'}) \leq \epsilon \quad \text{for all } m, m' \in [n, \eta(n)]. \]

- A sequence is metastable (with rate $E_{\bullet}$) if it is $[\epsilon, \eta]$-metastable (with bound $E_{\epsilon, \eta}$) for every sampling $\eta$ and all $\epsilon > 0$.

Remarks

- In general, metastability (with specified rate) is a relaxation of the Cauchy property by restricting to finite sub-tails of $(a_n)$.
- When no rates are specified, we have:
  - $(a_n)$ is metastable $\iff$ $(a_n)$ is a Cauchy sequence.
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A Uniform Metastability Principle (UMP)

Proposition (Uniform Metastability Principle [Duenez-I])

Let $T$ be a uniform theory in a signature $L$ such that:

- $L$ names a sort interpreted as a (discrete) linearly ordered set $(\mathbb{N}, <)$ elementarily extending $(\mathbb{N}, <)$ in models of $T$, and
- $L$ includes a symbol $a(\cdot)$ for a function $\mathbb{N} \to \mathbb{R}$.

Then, the following properties are equivalent:

1. All classical sequences $(a(n) : n \in \mathbb{N})$ obtained by interpreting $a(\cdot)$ in models of the theory $T$ are Cauchy.

2. There exists a collection $E_{\bullet} = (E_{\epsilon, \eta})$ of metastability rates that applies uniformly to all such sequences.

Furthermore, when these properties hold, the rate $E_{\bullet}$ depends only on the theory $T$. 
Some remarks about the UMP

- The UMP follows directly from the compactness theorem for first-order continuous logic.
- It holds for any logic for metric structures that is countably compact.
- \( \mathbb{N} \) can be replaced by any directed set (hence it holds for nets, rather than just sequences).
- It essentially states that metastable convergence with a prescribed rate is the only way to capture convergence in first-order continuous logic.

Moreover, the UMP implies the following metatheorem:

> “Whenever a theorem about convergence of sequences applies to a class of complete metric structures axiomatizable in continuous first-order logic, then the theorem admits a refinement as a statement about uniformly metastable convergence.”

We now switch to applications of this metatheorem.
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DCT structures (Dominated Convergence Theorem)

Let $T_{\text{DCT}}$ be the theory (in a suitable signature $L$) of all structures of the form $\mathcal{M} = (\mathbb{R}, (\mathbb{N}, <), (X, \mu), (\mathcal{L}^\infty(X), \int), \varphi\cdot)$, where

- $(\mathbb{N}, <)$ is the totally ordered set of natural numbers,
- $(X, \mu)$ is a finite measure space,
- $\varphi\cdot : \mathbb{N} \to B_1(\mathcal{L}^\infty(X))$ is a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in the unit ball of $\mathcal{L}^\infty(X)$.

**Definition**

A DCT structure is a countably saturated model $\mathcal{M} = (\mathbb{R}, (\mathbb{N}, <), (X, \mu), (\mathcal{L}^\infty(X), \int), \varphi\cdot)$ of $T_{\text{DCT}}$.

**Remark**

For all practical purposes (by proxy of a construction analogous to that of Loeb measure in nonstandard analysis) $\aleph_1$-saturation implies that the sort $(X, \mu, \int)$ of a DCT structure is a classical, countably additive probability space and $\int$ is classical integration of functions $f \in \mathcal{L}^\infty(X)$. 
Tao’s Metastable Dominated Convergence Theorem
Theorem (Dominated Convergence Theorem (DCT))

Let \( \mathcal{M} = (\mathbb{R}, (\mathbb{N}, <), (X, \mu), (\mathcal{L}_X, \int), \varphi_{\bullet}) \) be a DCT structure. If \((\varphi_n(x))\) is Cauchy for almost all \( x \in X \), then \((\int \varphi_n(x)d\mu(x))_{n \in \mathbb{N}}\) is Cauchy.

Since DCT structures are bona fide measure spaces endowed with classical integration, the usual proof of DCT applies.

Corollary (Metastable Dominated Convergence Theorem [Tao, 2008])

For every metastability rate \( E_{\bullet} \) there exists another metastability rate \( \tilde{E}_{\bullet} \) such that whenever \( E_{\bullet} \) is a metastability rate for the sequences \((\varphi_n(x))\) in \([-1,1]\), for almost all \( x \) in a finite measure space \((X, \mu)\), then \( \tilde{E}_{\bullet} \) is a metastability rate for \((\int \varphi_n(x)d\mu(x))\).

Proof.

Extend \( T_{\text{DCT}} \) to \( T' \) by adding the first-order axioms stating that \((\varphi_n(x))\) is \( E_{\bullet}\)-metastable for almost all \( x \). Every model \( \mathcal{M} \) of \( T' \) embeds into a (countably saturated) DCT structure for which DCT holds. By UMP, some metastability rate \( \tilde{E}_{\bullet} \) must apply to all sequences \((\int \varphi_n(x)d\mu(x))\).
Tao’s Metastable Dominated Convergence Theorem

Theorem (Dominated Convergence Theorem (DCT))

Let $\mathcal{M} = (\mathbb{R}, (\mathcal{N}, <), (X, \mu), (\mathcal{L}_X, \int), \varphi_\bullet)$ be a DCT structure. If $(\varphi_n(x))$ is Cauchy for almost all $x \in X$, then $(\int \varphi_n(x) d\mu(x))_{n \in \mathbb{N}}$ is Cauchy.

Since DCT structures are bona fide measure spaces endowed with classical integration, the usual proof of DCT applies.

Corollary (Metastable Dominated Convergence Theorem [Tao, 2008])

For every metastability rate $E_\bullet$ there exists another metastability rate $\widetilde{E}_\bullet$ such that whenever $E_\bullet$ is a metastability rate for the sequences $(\varphi_n(x))$ in $[-1, 1]$, for almost all $x$ in a finite measure space $(X, \mu)$, then $\widetilde{E}_\bullet$ is a metastability rate for $(\int \varphi_n(x) d\mu(x))$.

Proof.

Extend $T_{\text{DCT}}$ to $T'$ by adding the first-order axioms stating that $(\varphi_n(x))$ is $E_\bullet$-metastable for almost all $x$. Every model $\mathcal{M}$ of $T'$ embeds into a (countably saturated) DCT structure for which DCT holds. By UMP, some metastability rate $\widetilde{E}_\bullet$ must apply to all sequences $(\int \varphi_n(x) d\mu(x))$. \qed
Theorem (Mean Ergodic Theorem (von Neumann 1932))

Given a unitary transformation $U$ on a Hilbert space $H$ and a point $x \in H$, the sequence

$$AV_N(x) = \frac{1}{n} \sum_{k=0}^{N-1} U^n x \quad (n \in \mathbb{N})$$

of ergodic averages converges as $N \to \infty$.

Corollary (Metastable Mean Ergodic Theorem)

There exists a universal metastability rate $E_\bullet$ such that the sequence of ergodic averages $(AV_N(x))$ of any point $x$ in the unit ball of any Hilbert space $H$ under any unitary operator $U$ on $H$ is $E_\bullet$-metastable.
Proposition (Metastable Birkhoff ergodic theorem)

For every $\delta > 0$ there exists a rate $E^{(\delta)}$ such that if $T$ is measure-preserving on a probability space $(X, \mu)$, then for every measurable $f$ such that $\|f\|_{\infty} \leq 1$ there exists measurable $Y \subset X$ such that

- $\mu(X \setminus Y) \leq \delta$, and
- $\left(\frac{1}{n} \sum_{j<n} f \circ T^j(y)\right)_{n \in \mathbb{N}}$ converges pointwise with metastable rate $E^{(\delta)}$ for all $y \in Y$. 
Remarks

- Apart from the dependence on $\delta$, the rate $E^{(\delta)}$ is completely universal (independent of $(X, \mu, T)$).
  - This should be contrasted with the almost-uniform convergence implied by Egorov’s theorem, where the rates of uniform convergence depend not only on $\delta$ but also on the transformation $T$.
- In this formulation, it is necessary to impose a bound on $\|f\|_{\infty}$ (not merely on $\|f\|_{1}$).
As we have seen the Uniform Metastability Principle (UMP) is a consequence of the compactness of first-order continuous logic. In fact, it holds in any logic for metric structures that satisfies countable compactness.

**Question (Tao)**

*Is there a precise connection between the metastability and compactness?*
PC-classes

Let $\mathcal{C}$ be a class of $L$-structures.

$\mathcal{C}$ is said to be a **PC-class** if $\mathcal{C}$ can be axiomatized by a single sentence in some signature $L' \supseteq L$.

Equivalently, $\mathcal{C}$ is a PC-class if $\mathcal{C}$ can be axiomatized by an existential second-order $L$-sentence.

This definition applies to any logic: Given a logic $\mathcal{L}$ one can consider the PC-classes of $\mathcal{L}$.

**Fact**

If $\mathcal{L}$ is a compact logic, then the logic of existential second-order sentences of $\mathcal{L}$ inherits compactness from $\mathcal{L}$. 
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If $\mathcal{L}$ is a compact logic, then the logic of existential second-order sentences of $\mathcal{L}$ inherits compactness from $\mathcal{L}$. 
An $L$-structure $\mathcal{M}$ is $RPC_\Delta$-characterizable if there exists a predicate $R$ and an existential second-order $L \cup \{R\}$-theory $T$ such that the restriction of any model of $T$ to the predicate $R$ is isomorphic to $\mathcal{M}$. 
Theorem (X. Caicedo, E. Duenez, I)

Let $\mathcal{L}$ be a logic that is not countably compact. If $\mathcal{M}$ is any a metric structure of cardinality less than the first measurable cardinal, then $(\mathcal{M}, a)_{a \in \mathcal{M}}$ is $RPC_{\Delta}$-caracterizable in $\mathcal{L}$.

This allows us to show that

Uniform Metastability Principle $\iff$ Compactness.
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Let $\mathcal{L}$ be a logic that is not countably compact. If $\mathcal{M}$ is any a metric structure of cardinality less than the first measurable cardinal, then $(\mathcal{M}, a)_{a \in \mathcal{M}}$ is RPC\_\Delta-characterizable in $\mathcal{L}$.

This allows us to show that

\[
\text{Uniform Metastability Principle } \iff \text{Compactness.}
\]
More precisely, we have:

Corollary

Let \( \mathcal{L} \) be a logic.

1. If \( \mathcal{L} \) is countably compact, then \( \mathcal{L} \) satisfies UMP.
2. If \( \mathcal{L} \) is not countably compact, then UMP fails for \( RPC_{\Delta}(\mathcal{L}) \).
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Let $\mathcal{L}$ be a logic.

1. If $\mathcal{L}$ is countably compact, then $\mathcal{L}$ satisfies UMP.
2. If $\mathcal{L}$ is not countably compact, then UMP fails for $\text{RPC}_\Delta(\mathcal{L})$. 
Main References


