Categoricity questions in computable model theory

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A structure $\mathcal{A}$ is **computable** if $\mathcal{A}$ is finite or $|\mathcal{A}| = \mathbb{N}$ and all relations and operations in $\mathcal{A}$ are computable.
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Examples.

$(\mathbb{N}, +, \times)$ is computable;
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**Examples.**

- $(\mathbb{N}, +, \times)$ is computable;
- $(\mathbb{Q}, +, \times)$ has a computable presentation.
Computably categorical structures

- A computable structures $\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic if there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.
Computable Structures

Primitive recursive structures

Computably categorical structures

▶ A computable structures $\mathcal{A}$ and $\mathcal{B}$ are computably isomorphic if there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

▶ A computable structure $\mathcal{B}$ is computably categorical if $\mathcal{A}$ is computably isomorphic to every its computable presentation.

Examples.

▶ $(\mathbb{N}, +, \times)$ and $(\mathbb{Q}, +, \times)$ are computably categorical;

▶ $(\mathbb{Q}, <)$ is computably categorical;

▶ $(\mathbb{N}, <)$ is not computably categorical.
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- $(\mathbb{Q}, <)$ is computably categorical;
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Computably categorical structures

- An equivalence structure is computably categorical iff almost all equivalence classes have same size.
- A linear order is computably categorical iff it contains only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it contains only finitely many atoms.
- An abelian $p$-group is computably categorical iff it has the form $\bigoplus_i G_i$, where each $G_i$ is either $\mathbb{Z}_{p^n}$, or $\mathbb{Z}_{p^\infty}$, and $G_i$ are isomorphic to each other beginning some $i$.
- A torsion-free abelian group is computably categorical iff it is a subgroup of $\mathbb{Q}^n$. 

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Computable dimension

- (Goncharov). The **computable dimension** $\dim(A)$ is the least number $n \leq \omega$ for which there are computable presentations $A_1, A_2, \ldots, A_n$ such that every presentation of $A$ is computably isomorphic to one of $A_1, A_2, \ldots, A_n$. 

Example. $\dim(\mathbb{N}, <) = \omega$. 

Theorem (Goncharov). For every $n \leq \omega$ there is a computable structure $A$ such that $\dim(A) = n$. 

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**Example.** $\dim(\mathbb{N}, <) = \omega$.

**Theorem** (Goncharov). For every $n \leq \omega$ there is a computable structure $A$ such that $\dim(A) = n$. 
Computable structures \( A \) and \( B \) are \textbf{\( x \)-computably isomorphic} if there is an \( g \)-computable isomorphism from \( A \) onto \( B \).

\textbf{Examples.}

- \((N, <)\) is \(0'\)-computably categorical.
- A computable equivalence structure is \(0''\)-computably categorical.

\textbf{Theorem.} (Goncharov). If \( A \) is \(0'\)-computably categorical then either \( \dim(A) = 1 \), or \( \dim(A) = \omega \).
Computable structures $\mathcal{A}$ and $\mathcal{B}$ are **$x$-computably isomorphic** if there is an $g$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

A computable structure $\mathcal{B}$ is **$x$-computably categorical** if $\mathcal{A}$ is $x$-computably isomorphic to every its computable presentation.
**x**-computably categorical structures

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- $(\mathbb{N}, <)$ is $0'$-computably categorical.

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\textbf{Examples.}

- $(\mathbb{N}, <)$ is $0'$-computably categorical.

- A computable equivalence structure is $0''$-computably categorical.

\textbf{Theorem.} (Goncharov). If $\mathcal{A}$ is $0'$-computably categorical then either $\text{dim}(\mathcal{A}) = 1$, or $\text{dim}(\mathcal{A}) = \omega$. 
A computable structure $\mathcal{A}$ has a **degree of categoricity** $a$ if $a$ is the least Turing degree such that $\mathcal{A}$ is $x$-computably categorical.

Example. $(\mathbb{N}, <)$ has degree of categoricity $0'$. 

A degree of categoricity $a$ of a computable structure $\mathcal{A}$ is **strong** if $\mathcal{A}$ has two computable presentations $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $a \leq_T f$ for every isomorphism $f$ from $\mathcal{A}_0$ onto $\mathcal{A}_1$.

Question. Does every degree of categoricity is strong?
A computable structure \( A \) has a **degree of categoricity** \( a \) if \( a \) is the least Turing degree such that \( A \) is \( x \)-computably categorical.

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Degrees of Categoricity

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**Question.** Does every degree of categoricity is strong?
Non-strong degrees of categoricity

- (Csima, Stephenson). There is a computable rigid structure of computable dimension 3 whose degree of categoricity is not strong.

- (Bazhenov, K, Yamaleev). There is a computable rigid structure with the degree of categoricity $0'$ and this degree is not strong.

Both examples have the property that there are two pairs of copies $A_0, A_1$ such that the degree of categoricity is computable in $f_0 \oplus f_1$, where $f_0$ maps $A_0$ onto $A_1$ and $f_1$ maps $A_0$ onto $A_1$. 
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Both examples have the property that there are two pairs of copies $A_0^0, A_0^1$ and $A_0^0, A_1^1$ such that the degree of categoricity is computable in $f_0 \oplus f_1$, where $f_0$ maps $A_0^0$ onto $A_0^1$ and $f_1$ maps $A_1^0$ onto $A_1^1$. 
The categoricity spectrum of a computable structure $\mathcal{A}$ is a collection of all Turing degrees $\mathbf{x}$ such that $\mathcal{A}$ is $\mathbf{x}$-computably categorical.
The categoricity spectrum of a computable structure $\mathcal{A}$ is a collection of all Turing degrees $x$ such that $\mathcal{A}$ is $x$-computably categorical.

Or, equivalently

$$\text{CatSp}(\mathcal{A}) = \bigcap_{\mathcal{A}^0 \cong \mathcal{A}^1 \cong \mathcal{A}} \text{IsSp}(\mathcal{A}^0, \mathcal{A}^1),$$

where $\text{IsSp}(\mathcal{A}^0, \mathcal{A}^1)$ is the degrees computing at least one isomorphism from $\mathcal{A}^0$ onto $\mathcal{A}^1$. 
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The degree of categoricity of $\mathcal{A} =$ the least element of $\text{CatSp}(\mathcal{A})$. 
The spectral dimension $\text{SpDim}(\mathcal{A})$ of a computable structure $\mathcal{A}$ is the least $n \leq \omega$ such that

$$\text{CatSp}(\mathcal{A}) = \bigcap_{i < n} \text{IsSp}(\mathcal{A}^0_i, \mathcal{A}^1_i)$$

for some choice of computable presentations $\mathcal{A}^0_i \cong \mathcal{A}^1_i \cong \mathcal{A}$, $i < n$. 

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If $\mathcal{A}$ has a degree of categoricity $a$ then $\text{SpDim}(\mathcal{A})$ is the least $n \leq \omega$ such that $a \leq_T \bigoplus_{i<n} f_i$ for every $f_i \in \text{IsSp}(\mathcal{A}_i^0, \mathcal{A}_i^1)$, $i < n$. 
Properties of spectral dimension

- The structure $\mathcal{A}$ is computably categorical iff $\text{SpDim}(\mathcal{A}) = 0$. 
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- The structure $\mathcal{A}$ is computably categorical iff $\text{SpDim}(\mathcal{A}) = 0$.
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If $a$ is a degree of categoricity of $\mathcal{A}$ then $\text{SpDim}(\mathcal{A}) = 1$ iff $a$ is strong.
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- The structure $\mathcal{A}$ is computably categorical iff $\text{SpDim}(\mathcal{A}) = 0$.
- $\text{SpDim}(\mathcal{A}) \leq \text{Dim}(\mathcal{A}) - 1$.
- If $a$ is a degree of categoricity of $\mathcal{A}$ then $\text{SpDim}(\mathcal{A}) = 1$ iff $a$ is strong.
- For rigid structures $\text{SpDim}(\mathcal{A}) < \omega$ iff $\mathcal{A}$ has a degree of categoricity.
The structure $\mathcal{A}$ is computably categorical iff $\text{SpDim}(\mathcal{A}) = 0$.

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If $\mathbf{a}$ is a degree of categoricity of $\mathcal{A}$ then $\text{SpDim}(\mathcal{A}) = 1$ iff $\mathbf{a}$ is strong.

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There are rigid structures $\mathcal{A}$ without degree of categoricity (Fokina, Frolov, K). For such structures we have $\text{SpDim}(\mathcal{A}) = \omega$. 
Theorem. (Bazhenov, K., Yamaleev). For every $n < \omega$ there is a rigid computable structure of degree of categoricity $0'$ such that $\text{SpDim}(A) = n$.

Open Question. Is there a computable structure having a degree of categoricity such that $\text{SpDim}(A) = \omega$?

Note that if $a$ is a degree of categoricity of $A$ and $\text{SpDim}(A) < \omega$ then $a$ is the strong degree of categoricity of $A \oplus A \oplus \cdots \oplus A$.

Open Question. Does every degree of categoricity is a strong degree of categoricity of some other structure?
Degrees of categoricity with finite and infinite spectral dimension

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The proof from the computability theory point of view

For a (partial) function $R : \omega \times \omega \to \{0, 1\}$ and $G : \omega \to \omega$ we set

$$(R \diamond G)(x) = R(x, G(x))$$
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The proof consists of the construction of the construction of limitwise monotonic $G \equiv_T \emptyset'$ such that $G(x) < 2^n$ and

$$G \nleq_T (R_1 \diamond G) \oplus \cdots \oplus (R_{n-1} \diamond G)$$

for every $\{0, 1\}$-valued partial computable $R_1, \ldots, R_{n-1}$. 
A structure $\mathcal{A}$ is fully primitive recursive (f.p.r.) if $\mathcal{A}$ is finite or $|\mathcal{A}| = \mathbb{N}$ and all relations and operations in $\mathcal{A}$ are primitive recursive.
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An f.p.r. presentation of a countable structure $\mathcal{A}$ is any f.p.r. isomorphic copy of $\mathcal{A}$.

Examples.

- $(\mathbb{N}, +, \times)$ is f.p.r.;
- $(\mathbb{Q}, +, \times)$ has an f.p.r. presentation;
- $(\mathbb{N}, +_1, \mathcal{P}(x))$ has an f.p.r. presentation iff $\mathcal{P}(x)$ is primitive recursive.
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- $(\mathbb{N}, +, \times)$ is f.p.r.;
- $(\mathbb{Q}, +, \times)$ has an f.p.r. presentation;
- $(\mathbb{N}, +1, P(x))$ has an f.p.r. presentation iff $P(x)$ is primitive recursive.
Theorem. Every computable

- Equivalence structure (Cenzer, Remmel)
- Linear orders (Grigorieff)
- Torsion-free abelian groups (K., Melnikov, Ng)
- Boolean algebras (K., Melnikov, Ng)
- Abelian $p$-groups (K., Melnikov, Ng)

has an f.p.r. presentation.
Non-existence of f.p.r. presentations

**Theorem.** There are computable
- Torsion abelian groups (Cenzer, Remmel)
- Archimedean ordered abelian groups (K., Melnikov, Ng)
- Undirected graphs (K., Melnikov, Ng)
which have no f.p.r. presentations.
F.p.r. structures $\mathcal{A}$ and $\mathcal{B}$ are primitive recursively isomorphic if there is an isomorphism $\rho$ from $\mathcal{A}$ onto $\mathcal{B}$ such that both $\rho$ and $\rho^{-1}$ are primitively recursive.

Examples.

- The structure in empty signature is f.p.r. categorical.
- $(\mathbb{N}, +)$ is not f.p.r. categorical;
- $(\mathbb{Q}, \langle )$ is not f.p.r. categorical;

Question. Are there "nontrivial" examples of f.p.r. structures.
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Trivial f.p.r. categorical structures

Theorem. (K., Melnikov, Ng).

- A rigid relational structure is f.p.r. categorical iff it is finite.
Computable Structures
Primitive recursive structures

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- An equivalence structure is f.p.r. categorical iff either there finitely many classes at most one of which is infinite, or almost all equivalence classes have the size one.
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- A torsion-free abelian group $G$ is f.p.r. categorical iff $G \cong \{0\}$.
- An abelian $p$-group is f.p.r. categorical iff it has the form $\bigoplus_i \mathbb{Z}_{p^{n_i}}$, where $n_i = 1$ for almost all $i$. 
Non-trivial f.p.r. categorical structures

**Theorem.** (K., Melnikov, Ng). There is a rigid finitely generated f.p.r. categorical structure.
Non-trivial f.p.r. categorical structures

**Theorem.** (K., Melnikov, Ng). There is a rigid finitely generated f.p.r. categorical structure.

Let $A_f = \{ \langle x, y \rangle \in \mathbb{N}^2 : y < f(x) \}$; $L\langle x, y \rangle = \langle x + 1, 0 \rangle$; $R\langle x, y \rangle = (x, y + 1 \mod f(x))$. Then $A_f = (A_f, L, R)$ has a f.p.r. presentation iff $\text{graph}(f)$ is primitive recursive.
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To build a primitive recursive isomorphism $h_e$ from $B_e$ onto $A_f$ we will define

$$f(2^e(2m + 1)) \in \{2e + 1, 2e + 2\}$$
Non-trivial f.p.r. categorical structures

**Theorem.** (K., Melnikov, Ng). There is a rigid finitely generated f.p.r. categorical structure.

Let $A_f = \{\langle x, y \rangle \in \mathbb{N}^2 : y < f(x)\};$ $L\langle x, y \rangle = \langle x + 1, 0 \rangle;$ $R\langle x, y \rangle = (x, y + 1 \mod f(x))$. Then $A_f = (A_f, L, R)$ has a f.p.r. presentation iff $\text{graph}(f)$ is primitive recursive. To build a primitive recursive isomorphism $h_e$ from $B_e$ onto $A_f$ we will define

$$f(2^e(2m+1)) \in \{2e + 1, 2e + 2\}$$

Looking on $B_e$ we will increase the interval of $2e + 1$-values (on the arguments $2^e(2m + 1)$) to help to primitively recursively define $h_e(0), h_e(1), h_e(2), \ldots$. 
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Theorem. (K., Melnikov, Ng). There is a locally finite f.p.r. categorical structure which is not computably categorical.

Open Question. Is there is a non-trivial relational f.p.r. categorical structure?
Primitive recursive degrees

▶ $f \leq_{PR} g$ if $f$ can be derived from $g$ and primitive recursive functions by superposition and primitive recursion.
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- $a_{PR} \in D_{PR}$ is honest if $a_{PR}$ contains a function with primitive recursive graph.
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- **Claim**: the jump of honest $PR$-degree is again honest.
PR-degrees of categoricity

F.p.r. structures $A$ and $B$ are $\mathbf{x}_{PR}$-recursively isomorphic if there is a isomorphism $p$ from $A$ onto $B$ such that both $p \leq_{PR} \mathbf{x}_{PR}$ and $p^{-1} \leq_{PR} \mathbf{x}_{PR}$. 
F.p.r. structures $\mathcal{A}$ and $\mathcal{B}$ are $\mathbf{x}_{\text{PR}}$-recursively isomorphic if there is an isomorphism $\rho$ from $\mathcal{A}$ onto $\mathcal{B}$ such that both $\rho \leq_{\text{PR}} \mathbf{x}_{\text{PR}}$ and $\rho^{-1} \leq_{\text{PR}} \mathbf{x}_{\text{PR}}$.

An f.p.r. structure $\mathcal{B}$ is $\mathbf{x}_{\text{PR}}$-f.p.r. categorical if $\mathcal{A}$ is $\mathbf{x}_{\text{PR}}$-recursively isomorphic to every its f.p.r. presentation.
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- F.p.r. structures \( A \) and \( B \) are \( x_{PR} \)-recursively isomorphic if there is an isomorphism \( p \) from \( A \) onto \( B \) such that both \( p \leq_{PR} x_{PR} \) and \( p^{-1} \leq_{PR} x_{PR} \).

- An f.p.r. structure \( B \) is \( x_{PR} \)-f.p.r. categorical if \( A \) is \( x_{PR} \)-recursively isomorphic to every its f.p.r. presentation.

- An f.p.r. \( A \) has a PR-degree of categoricity \( a_{PR} \) if \( a_{PR} \) is the least Turing degree such that \( A \) is \( x \)-f.p.r. categorical.
Honest PR-degrees of categoricity

**Theorem.** (K., Melnikov, Ng). Every honest degree (in particular, \(0'_{PR}, 0''_{PR}, \ldots\)) is a PR-degree of categoricity of some f.p.r. structure.
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Suppose a function $g$ is given with primitive recursive graph. Then we can consider the function $f$ such that $f(2x + 1) = g(x)$, $f(2^{e+1}(2m + 1)) \in \{2e + 1, 2e + 2\}$, and the structure $A_f = (A_f, L, R)$, where

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Let $A_f^0$ be the standard f.p.r. presentation and let $A_f^1$ be the f.p.r presentation, where images of $\langle x, f(x) - 1 \rangle$ are primitive recursive on $x$. Then $g \leq_{PR} f \leq_{PR} h$ for every isomorphism $h$ from $A_f^1$ onto $A_f^0$. 
A preorder on f.p.r. presentations

- \( A \preceq_{pr} B \) if there is a primitive recursive isomorphism from \( A \) onto \( B \).
A preorder on f.p.r. presentations

- $\mathcal{A} \leq_{pr} \mathcal{B}$ if there is a primitive recursive isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.
- Let $P(\mathcal{A})$ be the class of all f.p.r. presentations of $\mathcal{A}$ modulo $\leq_{pr}$.

- $P(\mathbb{N}, +)$ has $\leq_{pr}$-least element but has no $\leq_{pr}$-greatest element. The same for every finitely generated f.p.r. structure.
- $P(\mathcal{Q}, <)$ has $\leq_{pr}$-greatest element but has no $\leq_{pr}$-least element.
- $P(\text{random graph})$ has no $\leq_{pr}$-greatest and $\leq_{pr}$-least elements.

Open Question. Can $P(\mathcal{A})$ have finite size $> 1$? If yes, which finite orderings are realizable?

Open Question. Is a structure $\mathcal{A}$ f.p.r categorical if and only if $P(\mathcal{A})$ forms a singleton?
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