Categoricity questions in computable model theory

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Examples.

- $(\mathbb{N}, +, \times)$ is computable;
- $(\mathbb{Q}, +, \times)$ has a computable presentation.

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Examples.

- $(\mathbb{N}, +, \times)$ and $(\mathbb{Q}, +, \times)$ are computably categorical;
- $(\mathbb{Q}, <)$ is computably categorical;
- $(\mathbb{N}, <)$ is not computably categorical.

Computably categorical structures

- An equivalence structure is computably categorical iff almost all equivalence classes have same size.
- A linear order is computably categorical iff it contains only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it contains only finitely many atoms.
- An abelian p-group is computably categorical iff it has the form $\bigoplus_i G_i$, where each G_i is either $\mathbb{Z}_{p^n}$, or $\mathbb{Z}_{p^\infty}$, and G_i are isomorphic to each other beginning some i.
- A torsion-free abelian group is computably categorical iff it is a subgroup of $\mathbb{Q}^n$.

Computable dimesion

- (Goncharov). The computable dimesion $\dim(\mathcal{A})$ is the least number $n \leq \omega$ for which there are computable presentations $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ such that every presentation of $\mathcal{A}$ is computably isomorphic to one of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$.

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Example. $\mathrm{dim}(\mathbb{N}, <) = \omega$.

Theorem (Goncharov). For every $n \leq \omega$ there is a computable structure $\mathcal{A}$ such that $\mathrm{dim}(\mathcal{A}) = n$.

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Theorem. (Goncharov). If $\mathcal{A}$ is $\mathbf{0}'$-computably categorical then either $\mathsf{dim}(\mathcal{A}) = 1$, or $\mathsf{dim}(\mathcal{A}) = \omega$.

Degrees of Categoricity

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- Example. $(\mathbb{N}, <)$ has degree of categoricity $\mathbf{0}'$.
- A degree of categoricity $\mathbf{a}$ of a computable structure $\mathcal{A}$ is strong if $\mathcal{A}$ has two computable presentations $\mathcal{A}^0$ and $\mathcal{A}^1$ such that $\mathbf{a} \leq_T f$ for every isomorphism f from $\mathcal{A}^0$ onto $\mathcal{A}^1$.

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Question. Does every degree of categoricity is strong?

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Both examples have the property that there are two pairs of copies $\mathcal{A}_0^0, \mathcal{A}_0^1$ and $\mathcal{A}_1^0, \mathcal{A}_1^1$ such that the degree of categoricity is computable in $f_0 \oplus f_1$, where f_0 maps $\mathcal{A}_0^0$ onto $\mathcal{A}_0^1$ and f_1 maps $\mathcal{A}_1^0$ onto $\mathcal{A}_1^1$.

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- The degree of categoricity of $\mathcal{A}$ = the least element of $\mathbf{CatSp}(A)$.

Spectral dimension

- The spectral dimension **SpDim**($\mathcal{A}$) of a computable structure $\mathcal{A}$ is the least $n \leq \omega$ such that

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  for some choice of computable presentations $\mathcal{A}_i^0 \cong \mathcal{A}_i^1 \cong \mathcal{A}$, $i < n$.
- If $\mathcal{A}$ has a degree of categoricity $\mathbf{a}$ then $\mathbf{SpDim}(\mathcal{A})$ is the least $n \leq \omega$ such that $\mathbf{a} \leq_T \bigoplus_{i<n} f_i$ for every $f_i \in \mathbf{IsSp}(\mathcal{A}_i^0, \mathcal{A}_i^1)$, $i < n$.

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- For rigid structures $\mathbf{SpDim}(\mathcal{A}) < \omega$ iff $\mathcal{A}$ has a degree of categoricity.
- There are rigid structures $\mathcal{A}$ without degree of categoricity (Fokina, Frolov, K). For such structures we have $\mathbf{SpDim}(\mathcal{A}) = \omega$.

Degrees of categoriciy with finite and infinite spectral dimension

Theorem. (Bazhenov, K., Yamaleev). For every $n < \omega$ there is a rigid computable structure of degree of categoricity $\mathbf{0}'$ such that $\mathbf{SpDim}(\mathcal{A}) = n$.

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Open Question. Is there a computable structure having a degree of categoricity such that $\mathbf{SpDim}(\mathcal{A}) = \omega$?

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Note that if $\mathbf{a}$ is a degree of categoricity of $\mathcal{A}$ and $\mathbf{SpDim}(\mathcal{A}) < \omega$ then $\mathbf{a}$ is the strong degree of categoricity of $\mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}$.

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Open Question. Does every degree of categoricity is a strong degree of categoricity of some other structure?

The proof from the computability theory point of view

- For a (partial) function $R : \omega \times \omega \to \{0, 1\}$ and $G : \omega \to \omega$ we set
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- If G and $R \diamond G$ are total and R is partially computable then $R \diamond G \leq_T G$.
- The proof consists of the construction of limitwise monotonic $G \equiv_T \emptyset'$ such that $G(x) < 2^n$ and
$$G \not\leq_T (R_1 \diamond G) \oplus \cdots \oplus (R_{n-1} \diamond G)$$
for every $\{0, 1\}$-valued partial computable $R_1, \ldots, R_{n-1}$.

Fully primitive recursive structures

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Examples.

- $(\mathbb{N}, +, \times)$ is f.p.r.;
- $(\mathbb{Q}, +, \times)$ has an f.p.r. presentation;
- $(\mathbb{N}, +1, P(x))$ has an f.p.r. presentation iff $P(x)$ is primitive recursive.

Existence of f.p.r. presentations

Theorem. Every computable

- Equivalence structure (Cenzer, Remmel)
- Linear orders (Grigorieff)
- Torsion-free abelian groups (K., Melnikov, Ng)
- Boolean algebras (K., Melnikov, Ng)
- Abelian p-groups (K., Melnikov, Ng)

has an f.p.r. presentation.

Non-existence of f.p.r. presentations

Theorem. There are computable

- Torsion abelian groups (Cenzer, Remmel)
- Archimedean ordered abelian groups (K., Melnikov, Ng)
- Undirected graphs (K., Melnikov, Ng)

which have no f.p.r. presentations.

F.p.r. categorical structures

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Question. Are there "nontrivial"examples of f.p.r. structures.

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Let $A_f = \{\langle x, y\rangle \in \mathbb{N}^2 : y < f(x)\}$; $L\langle x, y\rangle = \langle x+1, 0\rangle$; $R\langle x, y\rangle = (x, y+1 \mod f(x))$. Then $\mathcal{A}_f = (A_f, L, R)$ has a f.p.r. presentation iff $\mathsf{graph}(f)$ is primitive recursive.

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To build a primitive recursive isomorphism h_e from $\mathcal{B}_e$ onto $\mathcal{A}_f$ we will define

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Looking on $\mathcal{B}_e$ we will increase the interval of $2e+1$-values (on the arguments $2^e(2m+1)$) to help to primitively recursively define $h_e(0), h_e(1), h_e(2), \ldots$.

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Theorem. (K., Melnikov,Ng). There is a locally finite f.p.r. categorical structure which is not computably categorical.

Open Question. Is there is a non-trivial relational f.p.r. categorical structure?

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- Claim: the jump of honest PR-degree is again honest.

PR-degrees of categoricity

- F.p.r. structures $\mathcal{A}$ and $\mathcal{B}$ are $\mathbf{x}_{PR}$-recursively isomorphic if there is a isomorphism p from $\mathcal{A}$ onto $\mathcal{B}$ such that both $p \leq_{PR} \mathbf{x}_{PR}$ and $p^{-1} \leq_{PR} \mathbf{x}_{PR}$.

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Theorem. (K., Melnikov,Ng). Every honest degree (in particular, $\mathbf{0}'_{PR}, \mathbf{0}''_{PR}, \ldots$) is a PR-degree of categoricy of some f.p.r. structure.

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Suppose a function g is given with primitive recursive graph. The we can consider the function f such that $f(2x+1) = g(x)$, $f(2^{e+1}(2m+1)) \in \{2e+1, 2e+2\}$, and the structure
$\mathcal{A}_f = (A_f, L, R)$, where
$A_f = \{\langle x, y\rangle \in \mathbb{N}^2 : y < f(x)\}$; $L\langle x, y\rangle = \langle x+1, 0\rangle$;
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Let $\mathcal{A}^0_f$ be the standard f.p.r. presentation and let $\mathcal{A}^1_f$ be the f.p.r presentation, where images of $\langle x, f(x)-1\rangle$ are primitive recursive on x. Then $g \leq_{PR} f \leq_{PR} h$ for every isomorphism h from $\mathcal{A}^1_f$ onto $\mathcal{A}^0_f$.

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- Open Question. Is a structure $\mathcal{A}$ f.p.r categorical if and and only if $\mathbf{P}(\mathcal{A})$ forms a singleton?