On a problem of Mal’cev

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A structure is **computably categorical** if it has a unique computable copy, up to computable isomorphism.

Problem (Mal′cev, in the 1960-s)

Describe computably categorical abelian groups.

We have nice satisfactory classifications for:

- *p*-groups (Smith, indep. Goncharov)
- divisible torsion-free (Mal′cev) and arbitrary torsion-free (Nurtazin)
- infinite rank (Goncharov)

Missing cases:

- mixed of finite rank > 1
- **torsion**

If A is a torsion abelian group (TAG) then

$$A = \bigoplus_{p-\text{prime}} A_p.$$

How hard could this be?

Fact (M and Ng)

There exists a computable torsion A which is relatively c.c. but not c.c.

Thus, there is no **purely algebraic** description of c.c. TAG.

What would be considered a satisfactory classification of c.c. torsion groups?

Let $\mathcal{L}$ be the language of groups.

Recall that $\mathcal{L}^c_{\omega_1\omega}$ consists of computable infinitary $\mathcal{L}$-formulae.

We want a sentense $\Psi \in \mathcal{L}^c_{\omega_1\omega}$ such that, for a computable group G,

$$G \models \Psi \iff G \text{ is c.c. torsion abelian,}$$

and so that the syntactical complexity of Ψ is optimal.

- Each A_p must be of the form $\mathit{Finite} \oplus \bigoplus \mathbb{Z}_{p^\lambda}$ $(\lambda \in \omega \cup \{\infty\})$.
- We can produce a (crude) formula of complexity Π^c_5.
- Can we do better?

At this point the hard stuff begins. (Yes we can do better.)

So each A_p splits into a direct sum of cyclic and quasi-cyclic groups.

Given an equivalence structure $E = \sum_{i \in I} E_i$, define

$$G_E = \bigoplus_{i \in I} G_i,$$

where G_i is either cyclic or quasi-cyclic and p-order$(G_i) = \#E_i$ for each $i \in I$.

This is clearly uniformly effective.

Can we also uniformly pass from groups to eq.structures?

Let G be a direct sum of cyclic and quasi-cyclic groups,

$$G = \bigoplus G_i$$

Define E_G to be the equivalence structure in which the $i'th$ equivalence class E_i has size $=$ p-order(G_i).

Can we guess/produce the decomposition uniformly?

Proposition

The functor $G \to E_G$ is uniformly effective.

Furthermore, regardless of the Ulm type of the input abelian p-group G, the output of the uniform procedure is always an equivalence structure.

Proof: Induction within induction within induction, Kulikov-style.

It is convenient:

E.g., instead of checking $A \cong B$, check if $E_A \cong E_B$ (etc.)

Let us go back to c.c. torsion groups.

The algebraic condition below **cannot** capture categoricity (c.c.$\neq$ *r.c.c.*), but it will help.

Definition

We say that an abelian p-group G satisfies the weak homogeneity property (WHP) if it is either divisible, or for any $a \in G$ of order p and $h_p(a) < \infty$ there exist at most finitely many elements of order p and height $> h_p(a)$.

- **If A is c.c. torsion then a.e. A_p must satisfy the WHP.**
- Abelian p-groups satisfying the WHP admit a nice algebraic description (in particular, they are r.c.c.)
- The syntactical complexity of the WHP is (at most) Π^c_3.
- Saying $E_A \cong E_B$ is Π^0_2 for two p-groups satisfying the WHP.

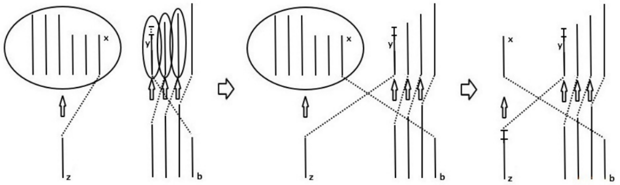
This pushes the complexity of Ψ down to $\Pi^c_4 \& \Sigma^c_4$.

If algebra won't help to simplify Ψ, then what would???

Idea: Try some uniform diagonalisation. If *A* is c.c. then the attempt must fail. Maybe it will give us some extra information.

Problem: Even the simplest possible diagonalisation attempt turns into something **brutal**.

The simplest possible strategy leads to a combinatorial mess:



We use various techniques from our previous works on equivalence structures to sort this out.

The sentence Ψ is the conjunction of the following formulae:

a. A is a TAG.
b. For very p, A_p is relatively c.c.
c. Γ describing the failure of the diagonalisation attempt (on A_{E_A}).
d. For every (partially) computable G,
if G is a TAG and there exists $k \in \omega$ such that:
   i. $\forall p > k$ G_p and A_p satisfy the WHP, and
   ii. $\forall p \leq k$ G_p and A_p are relatively c.c., and
   iii. $\forall p > k$ $E_{G_p} \cong E_{A_p}$, and
   iv. $\forall p \leq k$ $G_p \cong A_p$,

   then $G \cong_c A$.

In other words, Γ simplifies the WHP.

If Γ holds (Σ^0_3) then a.e. A_p satisfies the WHP (Σ^0_4).

Theorem (M. and Ng)

Computably categorical torsion abelian groups admit a Π^c_4 $\mathcal{L}^c_{\omega_1\omega}$-description.

Theorem (M. and Ng)

The index set of computably categorical torsion abelian groups is Π^0_4-complete. Thus, the complexity Π^c_4 above is **optimal**.

The proof of Π^0_4-hardness is a non-uniform infinite injury construction, but it is not that bad (compared to the stuff above).

Relatively c.c. structures admit a Σ_3^c-description.

C.c. torsion abelian groups are described by a Π_4^c-sentence.

And this is the best one can hope for.

I think it is a **positive (structure-type) result**.

It is easy to derive:

Theorem (M. and Ng)

If a TAG A is not c.c. then it has infinitely many computable copies, up to computable isomorphism.

This is new.

The only remaining case is “mixed of finite rank”.

From discrete groups to topological groups

Duality

Let $\mathbb{T}$ be the unit circle group.

Definition

The **dual group** of a topological group G is

$$\widehat{G} = \{\chi \mid \chi \text{ is a continuous group homomorphism from } G \text{ to } \mathbb{T}\}.$$

Theorem (Pontryagin - van Kampen)

Let G be a locally compact abelian group. Then $\widehat{G}$ is also locally compact abelian, and:

- $\widehat{\widehat{G}} \cong G$,
- G is compact separable iff $\widehat{G}$ is discrete countable.

Can we transfer computable results from discrete to topological groups? Well...

What is a computable Polish group?

Definition

A computable Polish group is a computable Polish (metric) space equipped with computable group operations.

Definition (Smith, can be traced to Nerode)

A profinite group is recursive if it is homeomorphic to the inverse limit of a computable surjective system of finite groups.

Theorem

(1) If G is computable discrete then $\widehat{G}$ is computable (compact) Polish.

(2) There exists a computable compact Polish group W s.t. the discrete $\widehat{W}$ has no computable copy.

The proof of (1) is **non-uniform** and uses some non-trivial stuff.

The group from (2) has several interesting features:

Corollary 1 (Answers a question of Willem Fouche)

The Haar measure on W is not computable.

Corollary 2

W is a profinite computable Polish group with no recursive copy.

Even the “half-effective” duality is useful.

We can list all partial computable Polish groups: $G_0, G_1, G_2, \ldots$.

Fact

$\{i : G_i$ is a compact Polish group $\}$ is Arithmetical (Π^0_3).

(This is not that obvious.)

Now it makes sense to measure the index set complexity of natural subclasses of compact groups.

If G is compact and H is the connected component of e (in G), then G/H is profinite.

Theorem

1. The index sets of **profinite** and of **connected compact** Polish groups are Arithmetical.
2. The topological isomorphism problems for **profinite (abelian) groups** and for **connected compact (abelian)** groups are Σ^1_1-complete.

- $\{i : G_i$ is a connected topological group$\}$ is Arithmetical.
- $\{(i, j) : G_i \cong G_j$ and G_i, G_j are connected$\}$ is Σ^1_1-complete.
- Thus, connected and profinite groups are **unclassifiable**.
- The proof uses definability analysis and the previous theorem.

Definition (Smith, after Nerode)

A profinite group is *recursive* if it is the limit of a computable surjective inverse system of finite groups.

($\widehat{G}$ stands for the Pontryagin dual of G.)

Theorem

- G is computable torsion iff $\widehat{G}$ is a recursive profinite group;
- G is computably categorical iff $\widehat{G}$ is computably categorical (as a recursive profinite group).

Corollary (follows from M. and Ng)

The index set of c.c. recursive profinite groups is Π^0_4-complete.

Eq.structures $\to$ torsion ab. groups $\to$ profinite ab.gr !!!

Thanks!