Reverse math, effectiveness and the Dual Ramsey theorem

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(with Damir Dzhafarov, Stephen Flood and Linda Brown Westrick)

Dual Ramsey theorem

An n-partition p of ω is a partition $\omega = B_0 \cup \cdots \cup B_{n-1}$ such that each partition block $B_i \neq \emptyset$. To make indexing unique, $\min(B_i) < \min(B_j)$ whenever $i < j$. Formally, $p : \omega \to n$ is onto and ordered.

- $(\omega)^n =$ the set of all n-partitions.

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- $(\omega)^n =$ the set of all n-partitions.

Note that $(\omega)^n \subseteq n^\omega$ and hence inherits the subspace topology from n^ω.

Let $(\omega)^n_{\text{fin}}$ be the set of all ordered $\sigma \in n^{<\omega}$ with $\text{range}(\sigma) = n$ and

$$[\sigma] = \{p \in (\omega)^n \mid \sigma \text{ is an initial segment of } p\}$$

These sets form a clopen basis for $(\omega)^n$.

An ω-partition p is a partition $\omega = B_0 \cup B_1 \cup \cdots$ into infinitely many nonempty blocks.

- $(\omega)^\omega$ = the set of all ω-partitions.

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Formally, $p \in (\omega)^\omega$ if $p : \omega \to \omega$ is ordered and onto.

A partition q is coarser than p is each p-block is contained in a q-block. That is, q if formed by collapsing p-blocks together.

- For $p \in (\omega)^\omega$, $(p)^n$ is the set of all $q \in (\omega)^n$ such that q is coarser than p.

Borel Dual Ramsey Theorem (Carlson and Simpson)

Let $(\omega)^n = C_0 \cup \cdots \cup C_{k-1}$ *be a Borel* k*-coloring of* $(\omega)^n$*. There is an* $i < k$ *and a partition* $p \in (\omega)^\omega$ *such that* $(p)^n \subseteq C_i$*.*

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Baire Dual Ramsey Theorem (Prömel and Voigt)

Let $(\omega)^n = C_0 \cup \cdots \cup C_{k-1}$ *be a* k*-coloring of* $(\omega)^n$ *with the Baire property. There is an* $i < k$ *and a partition* $p \in (\omega)^\omega$ *such that* $(p)^n \subseteq C_i$*.*

Coding the Dual Ramsey theorem in Z_2

In Z_2, $(\omega)^n$ is a class of 2nd order objects and we need a method to represent the colors $C_i \subseteq (\omega)^n$.

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Miller and Solomon, and Erhard studied effectiveness questions about variations of this lemma. Disadvantages of this approach:

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Miller and Solomon, and Erhard studied effectiveness questions about variations of this lemma. Disadvantages of this approach:

- The Carlson-Simpson lemma is iterated transfinitely in the proof of the Dual Ramsey theorem.
- The proof of the Carlson-Simpson lemma yields stronger results than the one needed for the Dual Ramsey theorem. The techniques of Miller and Solomon and of Erhard apply only to the stronger results.

Open Dual Ramsey theorem

A code for an open set is a set $O \subseteq \omega \times (\omega)^n_{\text{fin}}$. For $p \in (\omega)^n$, we say $p \in O$ if there is $\langle n, \sigma \rangle \in O$ such that $\sigma \prec p$.

An open k-coloring of $(\omega)^n$ is a sequence of open sets $\langle O_i : i < k \rangle$ such that $(\omega)^n = \cup_{i<k} O_i$.

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Open Dual Ramsey Theorem (Open-DRT^n_k)

For every open k-coloring $(\omega)^n = O_0 \cup \cdots \cup O_{k-1}$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $x \in (p)^n$, $x \in O_i$.

Formalizing a proof of Carlson and Simpson, Miller and Solomon showed that in RCA_0, Open-DRT$^{n+1}_k$ implies RT^n_k.

Borel Dual Ramsey theorem

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- If $\tau \in T$ is a leaf, then

$$
\begin{aligned}
n_\tau &= 0 \Rightarrow \tau \text{ codes } \emptyset \\
n_\tau &= 1 \Rightarrow \tau \text{ codes } (\omega)^n \\
n_\tau &= 2m+3 \Rightarrow \tau \text{ codes } \overline{[\sigma_m]} \\
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- If $\tau \neq \lambda$ is an interior node, then

$$n_\tau = 2m \Rightarrow \tau \text{ codes a union}$$
$$n_\tau = 2m+1 \Rightarrow \tau \text{ codes an intersection}$$

- λ codes the same set as $\langle m_T \rangle$.

Let T be a Borel code for a subset of $(\omega)^n$ and $p \in (\omega)^n$. An evaluation map for T at p is a function $f : T \to \{0,1\}$ which correctly says whether p is in the clopen sets coded at the leaves of T and correctly propagates down the tree at unions and intersections.

We say $p \in T$ (or $p \notin T$) if there is an evaluation function f for T at p such that $f(\lambda) = 1$ (or $f(\lambda) = 0$).

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A Borel k-coloring of $(\omega)^n$ is a sequence of Borel codes $\langle C_i : i < k \rangle$ such that $(\omega)^n = \cup_{i<k} C_i$.

Borel Dual Ramsey Theorem (Borel-DRT^k_n)

Let $(\omega)^n = C_0 \cup \cdots \cup C_{k-1}$ be a Borel k-coloring of $(\omega)^n$. There is an $i < k$ and a partition $p \in (\omega)^\omega$ such that $(p)^n \subseteq C_i$.

ATR_0 suffices to prove that evaluation maps exist, and ACA_0 proves they are unique, provided they exist.

Formalizing results in the setting of recursive mathematics, we proved

Theorem

Over RCA_0*, the statement "for every Borel code* B*, there is a point* p *such that* $p \in B$ *or* $p \notin B$*" implies* ATR_0*.*

Baire Dual Ramsey theorem

$M \subseteq (\omega)^n$ is meager if it is a countable union of nowhere dense sets.

$X \subseteq (\omega)^n$ has the Baire property if there is an open set O such that $X \triangle O$ is meager.

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The sets with the Baire property form a σ-algebra generated by the open sets and the meager sets.

If X has the Baire property, then X has a comeager approximation consisting of open sets U and V such that $U \cup V$ is dense and a sequence $\langle D_j : j \in \omega \rangle$ of dense open sets with the property that for all $p \in \cap_j D_j$,

$$p \in U \Rightarrow p \in X \qquad \text{and} \qquad p \in V \Rightarrow p \notin X$$

Let B be a Borel code. A Baire code for B consists of open sets U and V such that $U \cup V$ is dense and a sequence $\langle D_j : j \in \omega \rangle$ of dense open sets with the property that for all $p \in \cap_j D_j$,

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Let $(\omega)^n = C_0 \cup \cdots \cup C_{k-1}$ be a k-coloring of $(\omega)^n$ with the Baire property. There is an $i < k$ and a partition $p \in (\omega)^\omega$ such that $(p)^n \subseteq C_i$.

Baire Dual Ramsey Theorem (Baire-DRT^n_k)

For a Baire k-coloring of $(\omega)^n$, there is an $i < k$ and a partition $p \in (\omega)^\omega$ such that for all $x \in (p)^n$, $x \in O_i \cap \bigcap_j D_j$.

Combinatorial Dual Ramsey theorem

For a partition $x \in (\omega)^n$, the blocks are denoted $B^x_0, \ldots, B^x_{n-1}$.

Combinatorial Dual Ramsey Theorem (CDRT^n_k)

For every k-coloring $(\omega)^{n-1}_{fin} = C_0 \cup \cdots \cup C_{k-1}$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $x \in (p)^n$, $x \restriction min(B^x_{n-1}) \in C_i$.

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For every k-coloring $(\omega)^{n-1}_{fin} = C_0 \cup \cdots \cup C_{k-1}$, there is a partition $p \in (\omega)^\omega$ and a color i such that for all $x \in (p)^n$, $x \upharpoonright min(B^x_{n-1}) \in C_i$.

This combinatorial version implies the Carlson-Simpson lemma (with appropriate parameters) and is provable from transfinitely many applications of the Carlson-Simpson lemma.

RCA_0 proves CDRT^2_k since $(\omega)^1_{\mathrm{fin}} = \{0^i : i > 0\}$, so CDRT^2_k follows from RT^1_k.

Theorem (RCA_0)

- Baire-DRT^n_k, Open-DRT^n_k *and* $CDRT^n_k$ *are equivalent, and*
- Borel-DRT^n_k *implies these three principles.*

Therefore, RCA_0 *proves* Baire-DRT^2_k *and* Open-DRT^2_k

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- *Over* ATR_0, Baire-DRT^n_k *implies* Borel-DRT^n_k.
- ATR_0 *is equivalent to "every Borel code has a Baire code".*

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Theorem (RCA_0)

The following are equivalent

- $\mathsf{ATR}_0 +$ Baire-DRT^n_2
- *For every Borel code B for a subset of $(\omega)^n$, there is a partition $p \in (\omega)^\omega$ such that $(p)^n \subseteq B$ or $(p)^n \subseteq \overline{B}$.*

Theorem (RCA_0)

The statement "for every open set O in $(\omega)^3$, there is a $p \in (\omega)^\omega$ such that either $(p)^3 \subseteq O$ or $(p)^3 \subseteq \overline{O}$" implies ACA_0.

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Theorem (RCA_0)

The following are equivalent.

- ACA_0
- *Every closed subset of $(\omega)^3$ has a Baire code.*
- *Every open subset of $(\omega)^3$ has a Baire code.*

The exponent 3 is important here. If O is a computable open set in $(\omega)^2$, then there is a computable $p \in (\omega)^\omega$ such that either $(p)^2 \subseteq O$ or $(p)^2 \subseteq \overline{O}$.

Connection to Hindman's theorem

Hindman's Theorem for k-colorings

For every $c : \mathcal{P}_{fin}(\omega) \to k$, there is a set $X \subseteq \mathcal{P}_{fin}(\omega)$ and a color $i < k$ such that

- *X is closed under finite unions,*
- *X contains an infinite sequence of pairwise disjoint sets, and*
- *$c(F) = i$ for all $F \in X$.*

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Theorem (RCA_0)

The following are equivalent.

- *Hindman's Theorem for* k*-colorings.*
- CDRT^3_k *with the added restriction that the homogeneous partition* p *satisfy* $max(B_i^p) < min(B_{i+1}^p)$ *for all* $i > 0$*.*

Effective analysis: lower bounds for $n = 3$

The Borel codes for Σ^0_α and Π^0_α subsets of $(\omega)^3$ are defined inductively on $\alpha < \omega_1$.

- They are labelled trees (T, ℓ) with $T \subseteq \omega^{<\omega}$.
- For a leaf node τ, $\ell(\tau)$ is $\emptyset$, $(\omega)^3$, $[\sigma]$ or $\overline{[\sigma]}$ for $\sigma \in (\omega)^3_{\text{fin}}$.
- For an interior node τ, $\ell(\tau)$ is $\cup$ or $\cap$.

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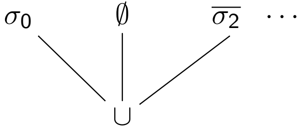
(T, ℓ) is a Borel code for a Σ^0_1 subset if T has height 1 and $\ell(\lambda) = \cup$. The code represents the set $\cup_{|\tau|=1} \ell(\tau)$.



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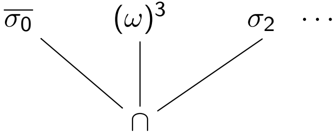

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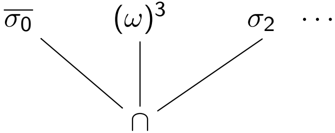

(T, ℓ) is a code for a Σ^0_α subset if $\ell(\lambda) = \cup$ and each labelled subtree above a node at level 1 is a code for a Σ^0_β or Π^0_β subset for some $\beta < \alpha$.

A code is computable if T and ℓ are computable.

Theorem

For every computable $\alpha > 0$, there is a computable code for a topologically Δ^0_α set $R \subseteq (\omega)^3$ such that every $p \in (\omega)^\omega$ homogeneous for $(\omega)^3 = R \cup \overline{R}$ computes $\emptyset^{(\alpha-1)}$ if $\alpha < \omega$ and computes $\emptyset^{(\alpha)}$ if $\alpha > \omega$.

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Step 1. Define an $\emptyset^{(\alpha)}$-computable clopen set $R \subseteq (\omega)^3$ such that if $p \in (\omega)^\omega$ is homogeneous for $(\omega)^3 = R \cup \overline{R}$, then $\emptyset^{(\alpha)} \leq_T p$.

(Solovay) For every $\alpha < \omega_1^{CK}$, there is a function $f_\alpha \equiv_T \emptyset^{(\alpha)}$ such that for every g dominating f, $\emptyset^{(\alpha)} \leq_T g$.

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Step 2. Effectively in the index of a $\emptyset^{(\alpha)}$-computable code for an open set $U \in (\omega)^3$, we can produce an index for a computable code for U as a topologically $\Sigma^0_{\alpha+1}$ (or Σ^0_α) set in the Borel hierarchy.

Effective analysis: upper bounds for $n = 2$

Theorem

Let R be a computable code for an open set in $(\omega)^2$. There is a computable $p \in (\omega)^\omega$ such that $(p)^2 \subseteq R$ or $(p)^2 \subseteq \overline{R}$.

The proof is necessarily non-uniform.

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Let R be a computable code for an open set in $(\omega)^2$. There is a computable $p \in (\omega)^\omega$ such that $(p)^2 \subseteq R$ or $(p)^2 \subseteq \overline{R}$.

The proof is necessarily non-uniform.

Theorem

For every coloring $(\omega)^2 = R \cup \overline{R}$ where R is a computable code for a Σ^0_{n+2} set, either there is a $\emptyset^{(n)}$-computable solution homogeneous for $\overline{R}$ or a $\emptyset^{(n+1)}$-computable solution homogeneous for R.

A coloring of $(\omega)^n$ is reduced if the color of p depends only on $p \upharpoonright \min(B^p_{n-1})$. Classically, reduced colorings are topologically open.

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Δ^0_n-rDRT^2_2 is the statement that every reduced 2-coloring of $(\omega)^2$ which is presented by a Δ^0_n code has a infinite homogeneous partition.

Theorem

For each $n \geq 2$:

- Δ^0_n-$\mathrm{rDRT}^2_2 \equiv_{sW} \mathrm{D}^n_2$
- *Over* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_{n-1}$, Δ^0_n-rDRT^2_2 *is equivalent to* D^n_2.

In particular, Δ^0_2-rDRT^2_2 *is equivalent to* SRT^2_2 *over* RCA_0.

D^n_2 says that every stable coloring $c : [\omega]^n \to 2$ has an infinite limit-homogeneous set.

Thank you!