VC_ℓ-dimension and growth rates of hereditary properties

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Model Theory and Counting

Claim: counting dichotomies and model theory go together.

Given a $\mathcal{L}$-theory T and a cardinal κ, let $I(T, \kappa)$ be the number of non-isomorphic models of T of cardinality κ.

Question

What are the possible behaviors of the function $I(T, \kappa)$?

• Model theoretic dividing lines were developed by Shelah to answer this question. One takeaway: the behavior of $I(T, \kappa)$ is related to structural properties of models of T.

• These dividing lines are a major organizing theme within modern model theory.

• Many model theoretic dividing lines can be characterized via counting dichotomies, e.g. stability, NIP, VC-minimality.

Hereditary Graph Properties

Notation: Given $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$.

A *hereditary graph property* is a class of graphs, $\mathcal{H}$, which is closed under isomorphism and induced subgraphs.

For a hereditary graph property $\mathcal{H}$, define

$$\mathcal{H}_n = \{\mathcal{M} \in \mathcal{H} : \mathcal{M} \text{ has domain } [n]\}$$

The *speed* of $\mathcal{H}$ is the function $n \mapsto |\mathcal{H}_n|$.

Hereditary Graph Properties

Question

What are the possible behaviors of the function $n \mapsto |\mathcal{H}_n|$?

Recall: every hereditary graph property $\mathcal{H}$ is axiomatized by a universal set of sentences $T_{\mathcal{H}}$.

This question is about counting finite models of a universal theory.

Cool Theorem

Question

What are the possible behaviors of the function $n \mapsto |\mathcal{H}_n|$?

It turns out there are discrete possibilities:

Theorem 1 ([3, 3, 5, 7, 2, 4])

If $\mathcal{H}$ is a hereditary graph property, then one of the following holds.

1. There is a polynomial $p(n)$ such that for large n, $|\mathcal{H}_n| = p(n)$.
2. There is $k \geq 2$ such that $|\mathcal{H}_n| = k^n(1 + o(1))$.
3. There exists $k > 1$ such that $|\mathcal{H}_n| = n^{(1-\frac{1}{k}+o(1))n}$,
4. There is an $\epsilon > 0$ such that $B_n \leq |\mathcal{H}_n| \leq 2^{n^{2-\epsilon}}$, where $B_n \sim (n/\log n)^n$.
5. There exists $k > 1$ such that $|\mathcal{H}_n| = 2^{(1-\frac{1}{k}+o(1))n^2/2}$ or $|\mathcal{H}_n| = 2^{\binom{n}{2}}$.

Connections to model theory

Question: How do these papers prove these results?

Answer: by giving structural characterizations of cases 1, 2, 3, 5.

Conclusion: Theorem 1 connects the behavior of $n \mapsto |\mathcal{H}_n|$ to structural properties of models of $T_{\mathcal{H}}$.

Claim

This should have something to do with model theory.

Connections to model theory

Claim

Theorem 1 should have something to do with model theory.

Problem

Generalize Theorem 1 to the setting of r-uniform hypergraphs for $r \geq 3$.

Goal: make progress on the problem using model theory.

Focus on jump to fastest speed

Theorem (Jump to fastest speed: graph case [2, 1, 6])

If $\mathcal{H}$ is a hereditary graph property, then either

(i) $|\mathcal{H}_n| = 2^{Cn^2(1+o(1))}$ *for some* $C > 0$, *or*

(ii) $|\mathcal{H}_n| \leq 2^{n^{2-\epsilon}}$ *for some* $\epsilon > 0$.

The ϵ in case (ii) comes from an argument using VC-dimension [2].

Theorem (Jump to fastest speed: hypergraph case [1, 6])

If $r \geq 3$ *and* $\mathcal{H}$ *is a hereditary property of* r*-uniform hypergraphs, then either*

(i) $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$ *for some* $C > 0$, *or*

(ii) $|\mathcal{H}_n| \leq 2^{o(n^r)}$.

Today: put ϵ in case (ii), using a generalization of VC-dimension.

VC_ℓ-dimension

Definition (boxes)

Suppose X is a set and $1 \leq \ell \leq r$. An (ℓ, r)*-box in* X is a set of the form

$$Y = Y_1 \ldots Y_\ell \subseteq X^r,$$

where $Y_1 \subseteq X^{k_1}, \ldots, Y_\ell \subseteq X^{k_\ell}$ for some $k_1 + \ldots + k_\ell = r$.

If $|Y_1| = \ldots = |Y_\ell| = m$, then Y has *height* m.

Note: if Y has height n, then $|Y| = m^\ell$.

Definition (VC_ℓ-dimension)

Given $\mathcal{F} \subseteq \mathcal{P}(X^r)$ and $1 \leq \ell \leq r$, the VC_ℓ-dimension of $\mathcal{F}$ is

$$\mathrm{VC}_\ell(\mathcal{F}) = \sup\{m \in \mathbb{N} : \mathcal{F} \text{ shatters an } (\ell, r)\text{-box of height } m \text{ in } X\}.$$

Sauer-Shelah for VC_ℓ-dimension

Fix $1 \leq \ell \leq r$ and $\mathcal{F} \subseteq \mathcal{P}(X^r)$.

Set $\pi_\ell(\mathcal{F}, m) = \sup\{|\mathcal{F} \cap A| : A$ is an (ℓ, r)-box of height m in $X\}$.

Observe: $\pi_\ell(\mathcal{F}, m) = 2^{m^\ell}$ if and only if $\mathrm{VC}_\ell(\mathcal{F}) \geq m$.

Theorem (Chernikov-Palacin-Takeuchi [9])

Suppose $1 \leq \ell \leq r$ and $\mathcal{F} \subseteq \mathcal{P}(X^r)$. If $\mathrm{VC}_\ell(\mathcal{F}) = d < \omega$, then there are $c = c(d)$ and $\epsilon = \epsilon(d) > 0$ such that for all m, $\pi_\ell(\mathcal{F}, m) \leq c2^{m^{\ell-\epsilon}}$.

Theorem (Sauer-Shelah)

Suppose $\mathcal{F} \subseteq \mathcal{P}(X)$. If $\mathrm{VC}_1(\mathcal{F}) = d < \omega$, then there is $c = c(d)$ such that for all m, $\pi_1(\mathcal{F}, m) \leq cm^d$.

VC_ℓ-dimension for hereditary $\mathcal{L}$-properties

Fix a fixed finite relational language, $\mathcal{L}$, with maximum arity $r \geq 1$.

A *hereditary $\mathcal{L}$-property* is a class of finite $\mathcal{L}$-structures which is closed under isomorphism and substructures.

Fix a non-trivial hereditary $\mathcal{L}$-property $\mathcal{H}$ (so $\mathcal{H}_n \neq \emptyset$ for all n).

Definition

Let $1 \leq \ell \leq r$ and let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula. Define

$$\mathcal{F}_\varphi(n) = \{U \subseteq [n]^{|\bar{x}|} : \text{ there is } \mathcal{M} \in \mathcal{H}_n \text{ such that } \varphi(\mathcal{M}) = U\}.$$

Then set

$$\mathrm{VC}_\ell(\varphi, \mathcal{H}) = \sup\{\mathrm{VC}_\ell(\mathcal{F}_\varphi(n)) : n \in \mathbb{N}\} \text{ and}$$

$$\mathrm{VC}_\ell(\mathcal{H}) = \sup\{\mathrm{VC}_\ell(\psi, \mathcal{F}) : \psi \text{ is quantifier-free}\}.$$

Bound in growth from finite VC_r-dimension

Theorem (T.)

If $\mathrm{VC}_r(\mathcal{H}) = d < \omega$, *then there is* $\epsilon = \epsilon(d) > 0$ *such that for all sufficiently large* n, $|\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$.

Proof of upper bound from finite VC$_r$-dimension

Proof.

Fix $\epsilon = \epsilon(d)$ and $C = C(d)$ from Chernikov-Palacin-Takeuchi.
Claim: For every relation $R(\bar{x})$ of $\mathcal{L}$, $|\mathcal{F}_R(n)| \leq C2^{n^{r-\epsilon}}$.

Proof.

If $|\bar{x}| < r$, then $\mathcal{F}_R(n) \subseteq \mathcal{P}([n]^{|\bar{x}|}) \Rightarrow |\mathcal{F}_R(n)| \leq 2^{n^{|\bar{x}|}} \leq 2^{n^{r-1}} \leq C2^{n^{r-\epsilon}}$.
If $|\bar{x}| = r$, then $\mathcal{F}_R(n) \subseteq \mathcal{P}([n]^r)$ and $\text{VC}_r(\mathcal{F}_R(n)) \leq d$ by assumption.
So Chernikov-Palacin-Takeuchi implies

$$C2^{n^{r-\epsilon}} \geq \pi_r(\mathcal{F}_R(n), n) = |\mathcal{F}_R(n)| \text{ (by definition).}$$

□

Thus we obtain the following bound:

$$|\mathcal{H}_n| \leq \prod_{R \text{ a relation of } \mathcal{L}} |\mathcal{F}_R(n)| \leq (C2^{n^{r-\epsilon}})^{|\mathcal{L}|} = C^{|\mathcal{L}|}2^{|\mathcal{L}|n^{r-\epsilon}} \leq 2^{n^{r-\epsilon/2}}.$$

Lower bound and the last jump

Theorem (lower bound)

If $1 \leq \ell \leq r$ *and* $\mathrm{VC}_\ell(\mathcal{H}) = \omega$*, then* $|\mathcal{H}_n| \geq 2^{\Omega(n^\ell)}$ *(i.e.* $|\mathcal{H}_n| \geq 2^{Cn^\ell}$ *for some* $C > 0$*).*

Theorem (Improved jump to fastest speed)

For all sufficiently large n, either

1. $|\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$ *for some* $C > 0$*.*
2. $|\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$ *for some* $\epsilon > 0$

Proof.

If $\mathrm{VC}_r(\mathcal{H}) = \infty$, then $|\mathcal{H}_n| \geq 2^{\Omega(n^r)} \Rightarrow |\mathcal{H}_n| = 2^{Cn^r(1+o(1))}$, some $C > 0$.
If $\mathrm{VC}_r(\mathcal{H}) < \infty$, then $|\mathcal{H}_n| \leq 2^{n^{r-\epsilon}}$ for some $\epsilon > 0$. □

VC_1-dimension characterizes polynomial growth

Theorem (T.)

If $VC_1(\mathcal{H}) = d < \omega$*, then there are* $c = c(d)$ *and* $k = k(d)$ *such that for sufficiently large* n*,* $|\mathcal{H}_n| \leq cn^k$*.*

Proof.

Fix $c = c(d)$ and $k = k(d)$ from Sauer-Shelah Lemma. Fix $R(\bar{x})$ a relation of $\mathcal{L}$. Then

$$|\mathcal{F}_R(n)| = \pi_1(\mathcal{F}_R(n), n^{|\bar{x}|}) \leq c(n^{|\bar{x}|})^k \leq cn^{rk}.$$

Thus $|\mathcal{H}_n| \leq \prod_{R\in\mathcal{L}} |\mathcal{F}_R(n)| \leq (cn^{rk})^{|\mathcal{L}|} = c^{|\mathcal{L}|} n^{kr|\mathcal{L}|}$. □

Corollary

For all sufficiently large n, either $|\mathcal{H}_n| \leq cn^k$ for some $c, k > 0$ or $|\mathcal{H}_n| \geq 2^{\Omega(n)}$.

Further Quesitons

We have shown the following.

Fact

If $\ell = 1$ or $\ell = r$, then $\mathrm{VC}_\ell(\mathcal{H}) = \infty \Leftrightarrow |\mathcal{H}_n| \geq 2^{\Omega(n^\ell)}$.

Question

Is it true that for any $1 \leq \ell \leq r$, $\mathrm{VC}_\ell(\mathcal{H}) = \infty \Leftrightarrow |\mathcal{H}_n| \geq 2^{\Omega(n^\ell)}$?

We know $\mathrm{VC}_\ell(\mathcal{H}) = \infty \Rightarrow |\mathcal{H}_n| \geq 2^{\Omega(n^\ell)}$ holds for any $1 \leq \ell \leq r$.

When $r = 2$, the answer is yes. (no $1 < \ell < r = 2$).

Answer

No!

Example

Definition

If (V, E) is a 3-uniform hypergraph and $xy \in \binom{V}{2}$, the *codegree* of xy is

$$d(xy) = \{z \in V : xyz \in E\}.$$

We say (V, E) has codegree (at most) k if for all $xy \in \binom{V}{2}$, $d(xy) = k$ (respectively $d(xy) \leq k$).

Let $\mathcal{L} = \{R(x, y, z)\}$ and let $\mathcal{H}$ be the class of finite 3-uniform hypergraphs of codegree at most 1.

Example

Claim

$\mathrm{VC}_2(\mathcal{H}) = 1 < \infty$

Proof.

It suffices to show $\mathrm{VC}_2(\mathcal{F}_R(n)) \leq 1$.

Let $Y = Y_1 Y_2$ be a $(2, 3)$-box of height 2 in $[n]^3$.

We may assume $Y_1 = \{a, b\} \subseteq [n]$ and $Y_2 = \{(c, d), (e, f)\} \subseteq [n]^2$.

Suppose $\mathcal{F}_R(n)$ shatters Y.

Then there is $\mathcal{M} \in \mathcal{H}_n$ such that $R(\mathcal{M}) = \{(a, c, d), (b, c, d)\}$.

But now cd has codegree at least 2, contradicting that $\mathcal{M} \in \mathcal{H}$. □

Example

Claim

$|\mathcal{H}_n| \geq 2^{\Omega(n^2)}$.

Proof.

A *Steiner triple system* is a 3-uniform hypergraph (V, E) such that every $xy \in \binom{V}{2}$ has codegree 1.

By [8, 10], if $n \equiv 1 \bmod 6$ or $n \equiv 3 \bmod 6$, we can choose a Steiner triple system, G_n with vertex set $[n]$.

If n is large, $e(G_n) = \binom{n}{2} / \binom{3}{2} \geq \frac{n^2}{7}$.

Thus, the number of sub-hypergraphs of G_n is at least $2^{e(G_n)} \geq 2^{\frac{n^2}{7}}$.

All sub-hypergraphs of G_n are in $\mathcal{H}_n$, so for all large $n \equiv 1 \bmod 6$ or $n \equiv 3 \bmod 6$, $|\mathcal{H}_n| \geq 2^{\frac{n^2}{7}}$. This implies $|\mathcal{H}_n| \geq 2^{\Omega(n^2)}$. □

Problem

We have shown $\mathcal{H}$ is a hereditary property of 3-uniform hypergraphs satisfying

- $\mathrm{VC}_2(\mathcal{H}) = 1 < \infty$ and
- $|\mathcal{H}_n| \geq 2^{\Omega(n^2)}$.

Thus when $r = 3$ and $\ell = 2$, $|\mathcal{H}_n| \geq 2^{\Omega(n^2)} \not\Rightarrow \mathrm{VC}_2(\mathcal{H}) = \infty$.

Problem

Given $1 < \ell < r$, characterize when $|\mathcal{H}_n| \geq 2^{\Omega(n^\ell)}$.

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