What do ultraproducts remember about the original models?

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McDuff constructed a family of separable II_1 factors $\mathcal{M}_\alpha$ for $\alpha \in 2^\omega$ so that when $\alpha \neq \beta$, $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are non-isomorphic.

Theorem (Boutonnet-Chifan-Ioana, 2015)

The McDuff factors are pairwise non-elementarily equivalent.

The proof begins by showing that if $\alpha(0) \neq \beta(0)$ then $M_\alpha \not\equiv M_\beta$.

Proof.

- Keisler-Shelah: M_α and M_β are elementarily equivalent iff there is an ultrafilter $\mathcal{U}$ so that $M_\alpha^\mathcal{U}$ and $M_\beta^\mathcal{U}$ are isomorphic.

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Proof.

- Keisler-Shelah: M_α and M_β are elementarily equivalent iff there is an ultrafilter $\mathcal{U}$ so that $M_\alpha^{\mathcal{U}}$ and $M_\beta^{\mathcal{U}}$ are isomorphic.
- For any ultrafilter $\mathcal{U}$, $M_\alpha^{\mathcal{U}}$ has *property* $\widetilde{V}$ iff $\alpha(0) = 1$.

$\square$

Definition

A non-separable von Neumann algebra $\mathcal{M}$ has property $\widetilde{V}$ if there exists a separable subalgebra $A \subseteq \mathcal{M}$ such that for any separable subalgebra $B \subseteq A' \cap \mathcal{M}$ and any separable subalgebra $C \subseteq \mathcal{M}$, there is a unitary $u \in \mathcal{M}$ such that $uBu^* \subseteq C' \cap \mathcal{M}$.

Question

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Whether $M_\alpha^{\mathcal{U}}$ satisfies $\widetilde{V}$ is purely a property of M_α—it does not depend on $\mathcal{U}$. Therefore we can ask:

Question

What property of M_α determines whether $M_\alpha^{\mathcal{U}}$ has property $\widetilde{V}$?

The ultraproduct construction takes an sequence of structures $\mathfrak{M}_i$ and a non-principal ultrafilter $\mathcal{U}$ and constructs a limiting object

$$\lim_{i \to \mathcal{U}} \mathfrak{M}_i = \mathfrak{M}^{\mathcal{U}}.$$

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Question

If we prove that some ultraproduct $\mathfrak{M}^{\mathcal{U}}$ has a property P (not necessarily elementary), what have we proven about the structures $\mathfrak{M}_i$?

Theorem (Łoś's Theorem)

When σ is a $\mathcal{L}$-formula, $\mathfrak{M}^{\mathcal{U}} \vDash \sigma$ iff for almost every i, $\mathfrak{M}_i \vDash \sigma$.

Theorem (Transfer Theorem)

For all integers x, y, let $\sigma_{x,y}$ be a sentence. Then

$$\text{For each } x \text{ there is an } y \text{ such that } \mathfrak{M}^{\mathcal{U}} \models \sigma_{x,y}$$

iff

$$\text{For each } x \text{ there is an } Y \text{ such that for almost every } i,$$

$$\mathfrak{M}_i \models \bigvee_{y \leq Y} \sigma_{x,y}.$$

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These are statements of the form

$$\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \sigma_{x,y}.$$

Most applications of ultraproducts outside of logic conclude by proving something of this form and then using transfer to produce a statement which doesn't mention the ultraproduct.

Theorem (ultraproduct proof by van den Dries–Schmidt)

For every n, d, k, there is a β so that for any field k and any system of equations

$$\begin{array}{cccc} f_{11}Y_1 & + \cdots + & f_{1\ell}Y_\ell & = f_1 \\ \vdots & & \vdots & \vdots \\ f_{k1}Y_1 & + \cdots + & f_{k\ell}Y_\ell & = f_k \end{array}$$

in $K[X_1, \ldots, X_n]$ with all functions having degree $\leq d$, there is a solution in $K[X_1, \ldots, X_n]$ iff there is a solution of degree $\leq \beta$.

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in $K[X_1, \ldots, X_n]$ with all functions having degree $\leq d$, there is a solution in $K[X_1, \ldots, X_n]$ iff there is a solution of degree $\leq \beta$.

Here we can work in the language of fields and $\sigma_{(n,d,k),\beta}$ is the sentence saying the statement above holds for the given parameters.

In the language of fields, we can quantify over polynomials given a fixed bound on the degree, but not over all functions. There is also a formula which holds iff a system of equations has a solution.

Theorem (Szemerédi's Theorem)

For every $\epsilon > 0$ and every k, there is an N so that whenever $A \subseteq [1, N]$ with $|A| \geq \epsilon N$, there exists an arithmetic progression

$$a, a + d, a + 2d, \ldots, a + kd \in A.$$

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Here we work with structures of the form $([1, N], A)$ with $A \subseteq [1, N]$, a partially defined addition operation, and extra predicates making it possible to compare sizes of sets.

$\sigma_{(\epsilon,k),d}$ is the sentence saying that if $|A| \geq \epsilon N$ then there is an arithmetic progression in A.

Theorem

Suppose that for all triples x, y, z we have a formula $\sigma_{x,y,z}$. Then

$$\text{For every } x \text{ there is a } y \text{ such that for every } z, \ \mathfrak{M}^{\mathcal{U}} \vDash \sigma_{x,y,z}$$

iff

$$\text{For every } x \text{ and } Z \text{ there is a } Y \text{ such that for almost every } i,$$

$$\mathfrak{M}_i \vDash \bigvee_{y \leq Y} \sigma_{x,y,Z(y)}.$$

Convergence is a "Π_3" statement

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The sequences $(t_n^{\mathfrak{M}_i})$ are *uniformly metastably convergent*:

> *for any* $\epsilon > 0$ *and any* $Z : \mathbb{N} \to \mathbb{N}$, *there is a value* N *so that, for each* i, *there is an* $n \leq N$ *with* $d(t_{Z(n)}^{\mathfrak{M}_i}, t_n^{\mathfrak{M}_i}) < \epsilon$.

Another example comes up in the context of fields. Consider the ultraproduct $K_{int}(X)$ of the fields $K_i(X)$. $K_{int}(X)$ is *freely generated* over the ultraproduct $K^{\mathcal{U}}(X)$.

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It turns out that actual proofs use the related fact:

> *Every finite set is contained in a finitely and freely generated subring.*

This is "Π_3": it means:

> *For every finite set S, there is a finite set T and a bound n so that:*
>
> - *every element of S is expressed by a rational polynomial of degree n in T, and*
> - *no element $t \in T$ is expressed by a rational polynomial in $T \setminus \{t\}$.*

This led us to:

Lemma (Simmons-T.)

For any b and any F, there is a bound Y so that whenever K is a ring and $S \subseteq K(X)$ with $|S| \leq b$, there is a $T \subseteq S$ with:

- *each $s \in S$ is a rational polynomial of elements $K(X, T)$ with degrees bounded by $y \leq Y$,*
- *if $t \in T$ then t is not a rational polynomial of elements in $K(X, T \setminus \{t\})$ with degrees bounded by $F(y)$.*

Suppose $\mathfrak{M}^{\mathcal{U}}$ satisfies a statement of the form

$$\forall x_1 \in \mathbb{N}\ \exists x_2 \in \mathbb{N}\ \cdots Qx_n \in \mathbb{N}\ \sigma_{x_1,\ldots,x_n}$$

where each $\sigma_{x_1,\ldots,x_n}$ is first-order.

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- we could view these as prenex statements in a modified logic,
- we stick to prenex statements for notational convenience,
- quantifiers over $\mathbb{N}$ could also include sets coded by $\mathbb{N}$, finite sequences, and so on.

We give an interpretation of

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Specifically, we will define

$$\{\mathfrak{M}_i\} \underset{i \to \infty}{\models} \forall x_1 \in \mathbb{N}\ \exists x_2 \in \mathbb{N}\ \cdots Q x_n \in \mathbb{N}\ \sigma_{x_1,\ldots,x_n}$$

directly using a modification of the ∀belard-∃loise formulation of the usual (Tarskian) semantics.

In order to define

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we'll use an alternate semantics, the *uniform semantics*.

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In the uniform semantics, ∀belard and ∃loise play in three *phases*:

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1. The *uniformizing phase*: before knowing which model $\mathfrak{M}_i$ they're going to play in, ∀belard and ∃loise make commitments about bounds on their plays,
2. The *selection phase*: they choose a model $\mathfrak{M}_i$ to play in,
3. The *bounded phase*: ∀belard and ∃loise play the usual Tarskian game in $\mathfrak{M}_i$, but with their numeric plays bounded by their commitments from the first phase.

In order to define $\{\mathfrak{M}_i\} \underset{i\to\infty}{\vDash} \forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \sigma_{x,y}$:

- In the uniformizing phase, first ∀belard picks a bound X, then ∃loise picks her bound Y, In order to define $\{\mathfrak{M}_i\} \underset{i\to\infty}{\models} \forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ \sigma_{x,y}$:

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- in the selection phase, ∃loise picks an I and ∀belard picks an $i > I$,
- in the bounded phase, ∀belard and ∃loise play the usual game in $\mathfrak{M}_i$ except that ∀belard must pick $x \leq X$ and ∃loise must pick $y \leq Y$.

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- in the selection phase, ∃loise picks I and ∀belard picks $i > I$,
- in the bounded phase, ∀belard picks $x \leq X$, then ∃loise picks $y \leq Y$, then ∀belard picks $z \leq \widetilde{Z}(y)$, and then they play the usual Tarskian game.

Definition

Write $\mathbb{S}_n^{\forall}$ for the set of ∀belard's strategies for "Π_n" sentences and $\mathbb{S}_n^{\exists}$ for the set of ∃loise's strategies for "Π_n" sentences.

We have seen:

- $\mathbb{S}_2^{\forall} = \mathbb{S}_2^{\exists} = \mathbb{N}$,
- $\mathbb{S}_3^{\forall} = \mathbb{N} \times (\mathbb{N} \to \mathbb{N})$, $\mathbb{S}_3^{\exists} = \mathbb{N}$.

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In general:

- $\mathbb{S}_{n+1}^\exists = \mathbb{S}_n^\forall$,
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- In the selection phase, ∃loise picks I and ∀belard picks $i > I$.
- In the bounding phase, ∀belard and ∃loise play the Tarskian game with numeric plays bounded by their strategies. After ∀belard chooses a number $x < X$, ∃loise chooses $y < Y, G' < G$, then they continue with the bounds given by $F(G')$, $G'(F(G'))$.

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Definition

$\{\mathfrak{M}_i\} \underset{i \to \infty}{\vDash} \sigma$ iff ∃loise has a winning strategy in the uniformized game.

Theorem

Let $\{\mathfrak{M}_i\}$ be a sequence of structures. Then the following are equivalent:

- $\{\mathfrak{M}_i\} \underset{i\to\infty}{\models} \sigma$ *(in the sense of the uniform semantics),*
- *every ultraproduct of the sequence $\{\mathfrak{M}_i\}$ satisfies σ.*

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- $\mathfrak{M}^{\mathcal{U}}$ fails to satisfy $\widetilde{V}$ if there is a $\gamma > 0$ and an m so that $\theta_m^{\mathfrak{M}} \geq \gamma$.

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Theorem (Goldbring-Hart-T.)

Suppose $\alpha, \beta \in 2^\omega$, n is least such that $\alpha(n) \neq \beta(n)$, and $\alpha(n) = 1$ while $\beta(n) = 0$. Then there is a formula $\theta_{m,n+1}$, an r, and an m (all given explicitly) so that

$$\theta_{m,n+1}^{\mathcal{M}_\alpha} = 0 \quad \text{and} \quad \theta_{m,n+1}^{\mathcal{M}_\beta} \geq r.$$

1. Finding effective bounds for proofs from ultraproducts.
    - Bounds on non-effective results from Banach space theory ("local unconditionality of the James space"),
    - bounds on results from algebra and differential algebra (testability of prime ideals, effective version of Ritt-Noetherianity) (with Simmons),
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2. Recognizing proofs which could be simplified using ultraproducts.
   - Hypergraph regularity and hypergraph analogs of quasirandomness,
   - Locality in finite model theory (with Lindell and Weinstein),
   - Fixed point theory in spaces with geodesic structure (Cho).

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- van den Berg, Briseid, and Safarik have a very similar interpretation for nonstandard analysis,
- Sanders has used this interpretation to analyze principles from analysis and reverse mathematics.

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Are there other semantics besides the uniform semantics? Are there other constructions like the ultraproduct which map these semantics to the Tarskian semantics?

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The end.