Theorem (Marks-U)

A disk in $\mathbb{R}^2$ can be partitioned into finitely many Borel sets, which can be translated to partition a square of the same area.
Let $a$ be an action of a group $\Gamma$ on a space $X$.

### Definition
We say subsets $A$ and $B$ of $X$ are $a$-equidecomposable if there are groups elements $\gamma_1, \ldots, \gamma_n$ in $\Gamma$ and a partition $\{A_1, \ldots, A_n\}$ of $A$ such that $\{\gamma_1 \cdot A_1, \ldots, \gamma_n \cdot A_n\}$ is a partition of $B$.

### Definition
A set $C$ is $a$-paradoxical if there is a partition $\{A, B\}$ of $C$ such that each of $A$ and $B$ is $a$-equidecomposable with $C$.

We say that the action is paradoxical if $X$ is $a$-paradoxical. The Banach Tarski paradox says that the ball in $\mathbb{R}^3$ is paradoxical by isometries.
Let $a$ be an action of a group $\Gamma$ on a space $X$.

**Definition**

We say that $a$ is amenable if there is a finitely additive probability measure on $X$ which is invariant under $a$.

We say that the group $\Gamma$ is amenable if the natural action of $\Gamma$ on itself by left translation is amenable.

An example: $\mathbb{Z}$ is amenable.

Fix a nonprincipal ultrafilter $U$ and for a set $A \subseteq \mathbb{Z}$ and define

$$
\mu(A) = \lim_U \frac{|A \cap [-n, n]|}{|[-n, n]|}
$$

**Theorem (Tarski)**

An action $a$ of a group $\Gamma$ on a set $X$ is not paradoxical if and only if it is amenable.
By contrast to $\mathbb{R}^3$, the isometry group of $\mathbb{R}^2$ is amenable. Using this one can show that there is a finitely additive, isometry invariant measure on all subsets of $\mathbb{R}^2$, which extends Lebesgue measure. This shows that the Banach-Tarski paradox is impossible in $\mathbb{R}^2$.

**Question (Tarski 1925)**

*Are a disk and a square in $\mathbb{R}^2$ with the same area equidecomposable by isometries?*

**Theorem (Dubins-Hirsch-Karush 1960’s)**

*Not possible using pieces whose boundaries are Jordan curves.*

**Theorem (Laczkovich 1990)**

*Yes!*
More generally

**Theorem (Laczkovich 1992)**

Suppose $k \geq 1$ and suppose $A, B \subseteq \mathbb{R}^k$ are bounded sets with the same positive Lebesgue measure, $\Delta(\partial A) < k$, and $\Delta(\partial B) < k$. Then $A$ and $B$ are equidecomposable by translations.

More recently

**Theorem (Grabowski, Máthé, and Pikhurko 2015)**

Laczkovich’s 1992 theorem with Lebesgue measurable or Baire measurable pieces.
**Theorem (Marks-U)**

Suppose \( k \geq 1 \) and suppose \( A, B \subseteq \mathbb{R}^k \) are bounded Borel sets with the same positive Lebesgue measure, \( \Delta(\partial A) < k \), and \( \Delta(\partial B) < k \). Then \( A \) and \( B \) are equidecomposable by translations using Borel pieces.
By translating and scaling our sets we can assume that we are working inside the torus $\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$, which we identify with $[0, 1)^k$.

Let $\bar{u} \in (\mathbb{T}^k)^d$ be a sequence of $d$ translations and let $a_{\bar{u}}$ be the natural action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ generated by $\bar{u}$:

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots n_d u_d + x$$
If there are a natural number $M$ and a Borel bijection $g : A \to B$ so that for all $x$ in $A$, there is $\gamma$ in $\mathbb{Z}^d$ with $\|\gamma\|_\infty \leq M$ such that $g(x) = \gamma \cdot x$, then $A$ and $B$ are equidecomposable by translations using Borel pieces.

For $\gamma$ in $\mathbb{Z}^d$ with $\|\gamma\|_\infty \leq M$, we have pieces $A_\gamma = \{x \in A \mid g(x) = \gamma \cdot x\}$.

For our theorem it is enough to find a sequence of translations $\bar{u}$ for which there are $g$ and $M$ as above.
We define a graph $G_{a\bar{u}}$ on $\mathbb{T}^k$ by putting an edge $\{x, y\}$ when there is some $\gamma \in \mathbb{Z}^d$ with $\|\gamma\|_\infty = 1$ so that $\gamma \cdot x = y$. 
For a rectangle $R$, we must understand $|A \cap R| - |B \cap R|$. Let $\lambda$ be Lebesgue measure on $\mathbb{T}^k$. If $F \subseteq \mathbb{T}^k$ is finite and $A \subseteq \mathbb{T}^k$ is $\lambda$-measurable, then the discrepancy of $F$ relative to $A$ is

$$D(F, A) = \frac{|F \cap A|}{|F|} - \lambda(A).$$

Note that if $D(R, A)$ and $D(R, B)$ are both small, then so is $\frac{|A \cap R| - |B \cap R|}{|R|}$. Laczkovich provides just the theorem we need to understand this discrepancy.
Suppose $G$ is a locally finite graph with vertex set $V$. We think of the edges of $G$ as forming a symmetric irreflexive relation on $V$.

For $f : V \to \mathbb{R}$, we define an $f$-flow on $G$ to be a real-valued function $\phi$ on the edges of $G$ such that for every edge $(x, y)$ in $G$, $\phi(x, y) = -\phi(y, x)$ and for every $x \in V$, 

$$f(x) = \sum_{y \in N(x)} \phi(x, y).$$

Suppose that $c$ is a nonnegative function on the edges of $G$ (where we may have $c(x, y) \neq c(y, x)$). We call $c$ a capacity function.

We say that an $f$-flow $\phi$ is bounded by $c$ if $\phi(x, y) \leq c(x, y)$ for every edge $(x, y)$ in $G$. We say that an $f$-flow $\phi$ is bounded if it is bounded by a constant capacity function.
Flows in locally finite graphs:

- For a locally finite graph it is easy to characterize when there is an f-flow bounded by some capacity function c.
- For finite graphs, this is a consequence of max flow min cut.
- For infinite graphs we can reduce to the countable case and use an ultralimit construction.
- Using the Ford-Fulkerson algorithm, if the function f and the capacity function c are integer valued and an f-flow satisfying c exists, then an integer valued flow exists.
Main steps of the proof.

1. Show that if there is a sequence of translations $\bar{u}$ such that $G_{a\bar{u}}$ has a bounded Borel $\chi_A - \chi_B$-flow which takes integer values, then there is a function $g$ which finishes the theorem.

2. Construct a real valued bounded Borel $\chi_A - \chi_B$-flow in $G_{a\bar{u}}$ where $\bar{u}$ is chosen using Laczkovich’s discrepancy estimates.

3. Convert the real valued flow to an integer valued one.
We construct a real valued Borel $\chi_A - \chi_B$-flow in $G_{a\bar{u}}$ by giving an explicit algorithm.

- Relies on Laczkovich’s discrepancy estimates for convergence.
- Uses the fact that the average of flows is a flow.
We show that any real-valued Borel $\chi_A - \chi_B$-flow in $G_{a\bar{u}}$ can be converted into an integer one which is close to the real valued one.

- Uses the integer version of a theorem about when flows exist.
- Uses work of Timár on the boundaries of finite sets in $\mathbb{Z}^d$.
- Uses very recent work of Gao, Jackson, Krohne and Seward on hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.

Finally we can use the integer valued flow to construct the bijection $g$ which moves points in $A$ at most some fixed distance in $G_{a\bar{u}}$. This finishes the proof.