Borel circle squaring

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Theorem (Marks-U)

A disk in $\mathbb{R}^2$ can be partitioned into finitely many Borel sets, which can be translated to partition a square of the same area.

Let a be an action of a group Γ on a space X.

Definition

We say subsets A and B of X are a-equidecomposible if there are groups elements $\gamma_1, \dots \gamma_n$ in Γ and a partition $\{A_1, \dots A_n\}$ of A such that $\{\gamma_1 \cdot A_1, \dots, \gamma_n \cdot A_n\}$ is a partition of B.

Definition

A set C is a-paradoxical if there is a partition $\{A, B\}$ of C such that each of A and B is a-equidecomposible with C.

We say that the action is paradoxical if X is a-paradoxical.
The Banach Tarski paradox says that the ball in $\mathbb{R}^3$ is paradoxical by isometries.

Let a be an action of a group Γ on a space X.

Definition

We say that a is amenable if there is a finitely additive probability measure on X which is invariant under a.

We say that the group Γ is amenable if the natural action of Γ on itself by left translation is amenable.
An example: $\mathbb{Z}$ is amenable.
Fix a nonprincipal ultrafilter U and for a set $A \subseteq \mathbb{Z}$ and define

$$\mu(A) = \lim_{U} \frac{|A \cap [-n, n]|}{|[-n, n]|}$$

Theorem (Tarski)

An action a of a group Γ on a set X is not paradoxical if and only if it is amenable.

By contrast to $\mathbb{R}^3$, the isometry group of $\mathbb{R}^2$ is amenable. Using this one can show that there is a finitely additive, isometry invariant measure on all subsets of $\mathbb{R}^2$, which extends Lebesgue measure. This shows that the Banach-Tarski paradox is impossible in $\mathbb{R}^2$.

Question (Tarski 1925)

Are a disk and a square in $\mathbb{R}^2$ with the same area equidecomposible by isometries?

Theorem (Dubins-Hirsch-Karush 1960's)

Not possible using pieces whose boundaries are Jordan curves.

Theorem (Laczkovich 1990)

Yes!

More generally

Theorem (Laczkovich 1992)

Suppose $k \geq 1$ and suppose $A, B \subseteq \mathbb{R}^k$ are bounded sets with the same positive Lebesgue measure, $\Delta(\partial A) < k$, and $\Delta(\partial B) < k$. Then A and B are equidecomposable by translations.

More recently

Theorem (Grabowski, Máthé, and Pikhurko 2015)

Laczkovich's 1992 theorem with Lebesgue measurable or Baire measurable pieces.

Theorem (Marks-U)

Suppose $k \geq 1$ *and suppose* $A, B \subseteq \mathbb{R}^k$ *are bounded Borel sets with the same positive Lebesgue measure,* $\Delta(\partial A) < k$*, and* $\Delta(\partial B) < k$*. Then* A *and* B *are equidecomposable by translations using Borel pieces.*

By translating and scaling our sets we can assume that we are working inside the torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, which we identify with $[0,1)^k$.
Let $\bar{u} \in (\mathbb{T}^k)^d$ be a sequence of d translations and let $a_{\bar{u}}$ be the natural action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ generated by $\bar{u}$:

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots n_d u_d + x$$

If there are a natural number M and a Borel bijection $g : A \to B$ so that for all x in A, there is γ in $\mathbb{Z}^d$ with $\|\gamma\|_\infty \leq M$ such that $g(x) = \gamma \cdot x$, then A and B are equidecomposible by translations using Borel pieces.
For γ in $\mathbb{Z}^d$ with $\|\gamma\|_\infty \leq M$, we have pieces $A_\gamma = \{x \in A \mid g(x) = \gamma \cdot x\}$.
For our theorem it is enough to find a sequence of translations $\bar{u}$ for which there are g and M as above.

We define a graph $G_{a_{\bar{u}}}$ on $\mathbb{T}^k$ by putting an edge $\{x, y\}$ when there is some $\gamma \in \mathbb{Z}^d$ with $\|\gamma\|_\infty = 1$ so that $\gamma \cdot x = y$.

For a rectangle R, we must understand $|A \cap R| - |B \cap R|$.
Let λ be Lebesgue measure on $\mathbb{T}^k$.
If $F \subseteq \mathbb{T}^k$ is finite and $A \subseteq \mathbb{T}^k$ is λ-measurable, then the *discrepancy of F relative to A* is

$$D(F, A) = \left| |F \cap A| / |F| - \lambda(A) \right|.$$

Note that if $D(R, A)$ and $D(R, B)$ are both small, then so is $\frac{|A \cap R| - |B \cap R|}{|R|}$.
Laczkovich provides just the theorem we need to understand this discrepancy.

Suppose G is a locally finite graph with vertex set V. We think of the edges of G as forming a symmetric irreflexive relation on V.

For $f : V \to \mathbb{R}$, we define an f-flow on G to be a real-valued function ϕ on the edges of G such that for every edge (x, y) in G, $\phi(x, y) = -\phi(y, x)$ and for every $x \in V$,

$$f(x) = \sum_{y \in N(x)} \phi(x, y).$$

Suppose that c is a nonnegative function on the edges of G (where we may have $c(x, y) \neq c(y, x)$). We call c a capacity function.

We say that an f-flow ϕ *is bounded by* c if $\phi(x, y) \leq c(x, y)$ for every edge (x, y) in G. We say that an f-flow ϕ is *bounded* if it is bounded by a constant capacity function.

Flows in locally finite graphs:

- For a locally finite graph it is easy to characterize when there is an f-flow bounded by some capacity function c.
- For finite graphs, this is a consequence of max flow min cut.
- For infinite graphs we can reduce to the countable case and use an ultralimit construction.
- Using the Ford-Fulkerson algorithm, if the function f and the capacity function c are integer valued and an f-flow satisfying c exists, then an integer valued flow exists.

Main steps of the proof.

1. Show that if there is a sequence of translations $\bar{u}$ such that $G_{a_{\bar{u}}}$ has a bounded Borel $\chi_A - \chi_B$-flow which takes integer values, then there is a function g which finishes the theorem.
2. Construct a *real valued* bounded Borel $\chi_A - \chi_B$-flow in $G_{a_{\bar{u}}}$ where $\bar{u}$ is chosen using Laczkovich's discrepancy estimates.
3. Convert the real valued flow to an integer valued one.

We construct a real valued Borel $\chi_A - \chi_B$-flow in $G_{a_{\bar{u}}}$ by giving an explicit algorithm.

- Relies on Laczkovich's discrepancy estimates for convergence.
- Uses the fact that the average of flows is a flow.

We show that any real-valued Borel $\chi_A - \chi_B$-flow in $G_{a_{\bar{u}}}$ can be converted into an integer one which is close to the real valued one.

- Uses the integer version of a theorem about when flows exist.
- Uses work of Timár on the boundaries of finite sets in $\mathbb{Z}^d$.
- Uses very recent work of Gao, Jackson, Krohne and Seward on hyperfiniteness of free Borel actions of $\mathbb{Z}^d$.

Finally we can use the integer valued flow to construct the bijection g which moves points in A at most some fixed distance in $G_{a_{\bar{u}}}$. This finishes the proof.