Fraïssé Theory for C^*-algebras

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1. Introduction
2. $\mathcal{Z}$ and $\mathcal{W}$
3. Connections with quantifier elimination

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The first two slides of this talk can be taken from Bradd's talk. In particular, we will work in the setting of C^*-algebras.

Definition

A trace τ on a C^*-algebra A is a linear functional $\tau\colon A \to \mathbb{C}$ with $\|\tau\| = 1$, $\tau(aa^*) \geq 0$ and $\tau(ab) = \tau(ba)$ for all $a, b \in A$. τ is faithful if $\tau(aa^*) = 0$ implies $a = 0$.

If A and B are C^*-algebras, σ, τ traces on A and B resp., and ϕ is such that $\tau(\phi(a)) = \sigma(a)$ for all $a \in A$, ϕ is said trace preserving, denoted $\phi\colon (A, \sigma) \to (B, \tau)$. If A and B are unital the pullback of a trace is a trace, so every unital embedding is trace preserving for some traces.

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If we consider the language of C^*-algebra together with a trace, trace preserving injections are exactly our embeddings.

Let $\mathcal{L}$ be a language including the language of C^*-algebras, and $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures considered with distinguished generators. $\mathcal{K}$ is said a Fraïssé class if it satisfies:

- the JEP: for all $A, B \in \mathcal{K}$ there is $C \in \mathcal{K}$ in which A and B both embed;
- the NAP: if A, B_1, B_2 are in $\mathcal{K}$ and $\phi_i \colon A \to B_i$ are $\mathcal{L}$-embeddings, $F \subseteq A$ is finite and $\epsilon > 0$ then there are $C \in \mathcal{K}$ and $\psi_i \colon B_i \to C$ such that

$$\|\psi_1 \circ \phi_1(a) - \psi_2 \circ \phi_2(a)\| < \epsilon, \text{ whenever } a \in F$$

Let $\mathcal{K}_n$ be the space of all n-generated elements of $\mathcal{K}$, $A, B \in \mathcal{K}_n$ with distinguished generators $\bar{a}$ and $\bar{b}$. Consider the pseudo-metric

$$d_n(A, B) = \inf_{C,\phi,\psi} \sup_{i \leq n} \left\|\phi(\bar{a}) - \psi(\bar{b})\right\|_C$$

where C is quantified in $\mathcal{K}$ and ψ, ϕ quantify over all embeddings $\phi \colon A \to C$, $\psi \colon B \to C$.

- the WPP: the $\mathcal{K}_n$ considered with the pseudo-metric d_n is separable, for all n.

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If $\mathcal{K}$ is a Fraïssé class, and M is a limit of structures in $\mathcal{K}$, M is called a $\mathcal{K}$-structure. A $\mathcal{K}$-structure is

- $\mathcal{K}$-universal if every element of $\mathcal{K}$ embeds in M
- approximately $\mathcal{K}$-homogeneous if for all $A \in \mathcal{K}$, $F \subseteq A$ finite, $\epsilon > 0$ and $\phi_1, \phi_2 \colon A \to M$ embeddings there is an automorphism ρ of M with
$$\|\rho(\phi_1(a)) - \phi_2(a)\| < \epsilon, \; a \in F$$

If $\mathcal{K}$ is a Fraïssé class and M is a separable $\mathcal{K}$-structure which is both $\mathcal{K}$-universal and approximately $\mathcal{K}$-homogeneous, M is said a Fraïssé limit.

Theorem (Ben Yaacov)

If $\mathcal{K}$ is a Fraïssé class and its Fraïssé limit exists, it is unique up to isomorphism.

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If $\mathcal{K}$ is a Fraïssé class and its Fraïssé limit exists, it is unique up to isomorphism.

If p and q are coprime, let

$$Z_{p,q} = \{f \in C([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes 1,\ f(1) \in 1 \otimes M_q\}.$$

$Z_{p,q}$ are called dimension drop algebras. Traces of $Z_{p,q}$ correspond to probability measures on $[0,1]$ by

$$\tau_\mu(f) = \int_0^1 \tau(f(t))d\mu(t).$$

All traces are of this form. If μ is diffuse, τ is said diffuse. Every (nonzero) *-homomorphism $\phi\colon Z_{p,q} \to Z_{p',q'}$ is trace preserving for some traces σ and τ.

Proposition

- *pq divides $p'q'$ if and only if there is an embedding $Z_{p,q} \to Z_{p',q'}$.*
- *Let σ, τ be faithful traces on $Z_{p,q}$. If τ is diffuse, there is an embedding $(Z_{p,q}, \sigma) \to (Z_{p,q}, \tau)$.*
- *If also σ is diffuse, ϕ can be chosen to be an isomorphism.*

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Theorem (Jiang-Su)

There are increasing sequences of coprimes p_i, q_i and maps $\phi_i \colon Z_{p_i,q_i} \to Z_{p_{i+1},q_{i+1}}$ such that $\mathcal{Z} = \lim_i(Z_{p_i,q_i}, \phi_i)$ is simple, monotracial and has the same K-theory as $\mathbb{C}$. Let p_i, q_i and ψ_i and $\mathcal{A} = \lim(Z_{p_i,q_i}, \psi_i)$. If $\mathcal{A}$ is simple monotracial and has the same K-theory as $\mathcal{Z}$, then $\mathcal{A} \cong \mathcal{Z}$.

Jiang and Su's $\mathcal{Z}$ is pivotal in the classification programme of C^*-algebras. $\mathcal{Z}$ is unique and universal in many senses. It is self-absorbing ($\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$) in a very strong sense. The (amenable) algebras that have the property of absorbing $\mathcal{Z}$ (i.e., $A \otimes \mathcal{Z} \cong A$) are the ones for which there are hopes of obtaining classification.

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Theorem (EFHKKL, Masumoto)

The class $\{(Z_{p,q}, \tau) \mid p, q \text{ coprimes}, \tau \text{ faithful trace}\}$ *is a Fraïssé class.* $\mathcal{Z}$ *is its Fraïssé limit.*

So for all p, q *coprimes,* $\phi_1, \phi_2 \colon Z_{p,q} \to \mathcal{Z}$, $F \subseteq Z_{p,q}$ *and* $\epsilon > 0$ *there is an automorphism* ρ *of* $\mathcal{Z}$ *such that*

$$\|\rho(\phi_1(a)) - \phi_2(a)\| < \epsilon$$

for all $a \in F$.

EFHKKL's Proof is based on some known facts about $\mathcal{Z}$. Masumoto's one on a careful study of what the maps between dimension drop algebras can be.

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Question

As we know that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$, what can we say about maps $\phi_1, \phi_2 \colon Z_{p,q} \otimes Z_{p',q'} \to \mathcal{Z}$? In other terms, can we prove that $\mathcal{Z}$ and $\mathcal{Z} \otimes \mathcal{Z}$ are the Fraïssé limit of the same class?

This is more difficult than one can think. In fact, maps $Z_{p,q} \otimes Z_{p,q} \to Z_{p',q'} \otimes Z_{p',q'}$ are more complicated than one can think. Despite that, maps $Z_{p,q} \otimes Z_{p',q'} \to Z_{p'',q''}$ are always well behaved.

Theorem (Jacelon-V.)

The class $\{(Z_{p,q}, \tau), (Z_{p,q} \otimes Z_{p',q'}, \sigma)\}$ is a Fraïssé class. Both $\mathcal{Z}$ and $\mathcal{Z} \otimes \mathcal{Z}$ are its Fraïssé limits, so $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$.

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Let

$$A_{n,k} = C([0,1], M_n \otimes M_k) \mid f(0) = 1 \otimes c,\ f(1) = 1_{n-1} \otimes c\}$$

These are called Razak's blocks. Traces are, as before, given by probability measures on the interval. The absence of the unit doesn't allow to say that every embedding of $A_{n,k} \to A_{n',k'}$ is trace preserving.

Proposition

- *If σ, τ are faithful diffuse traces on $A_{n,k}$ then there is an isomorphism $(A_{n,k}, \sigma) \to (A_{n,k}, \tau)$*
- *If $p \geq 2$ and τ, σ are faithful traces on $A_{n,k}$ and $A_{pn,(pn-1)k}$, τ being diffuse, there is a trace preserving embedding $\phi \colon (A_{n,k}, \sigma) \to (A_{pn,(pn-1)k}, \tau)$.*

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Theorem (Jacelon)

There are increasing sequences n_i, k_i and trace preserving (for some traces) maps $\phi_i \colon A_{n_i,k_i} \to A_{n_{i+1},k_{i+1}}$ such that $\mathcal{W} = \lim_i(A_{n_i,k_i}, \phi_i)$ is simple, stably projectionless monotracial and with trivial K-theory.

Let n_i, k_i and ψ_i and $A = \lim(A_{n_i,k_i}, \psi_i)$. If A is simple monotracial stably projectionless and with trivial K-theory, then $A \cong \mathcal{W}$.

Recent work of Elliott-Niu and Gong-Lin showed the first evidences that $\mathcal{W}$ plays the same role in the classification of nonunital algebras as $\mathcal{Z}$ does for the unital case. $\mathcal{W}$ is a universal objects in many ways. On the other hand, that $\mathcal{W} \otimes \mathcal{W} \cong \mathcal{W}$ was only recently proved, involving a long and complicated proof in classification theory.

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Theorem (Jacelon-V.)

The class $\{(A_{n,k}, \sigma) \mid \sigma$ *is a faithful trace*$\}$ *is a Fraïssé class.* $\mathcal{W}$ *is its limit.*

Whenever $\phi_1, \phi_2 \colon (A_{n,k}, \sigma) \to (\mathcal{W}, \tau)$ *are embeddings for some faithful diffuse* σ *and* $F \subseteq A_{n,k}$ *and* $\epsilon > 0$ *are given there is an automorphism* ρ *of* $\mathcal{W}$ *such that*

$$\|\rho(\phi_1(a)) - \phi_2(a)\| < \epsilon$$

for $a \in F$.

The techniques used are similar to the ones of Masumoto's proof that $\mathcal{Z}$ is a Fraïssé limit.

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Proposition

Fix n, k.

- *There are well-behaved maps $A_{n,k}\to A_{n',k'}$ for some n', k'.*
- *There are well-behaved maps $A_{n,k}\otimes A_{n,k}\to A_{n',k'}$ for some n', k'.*

Problem

Not all maps $A_{n,k}\otimes A_{n,k}\to A_{n',k'}$ are well-behaved. Proving NAP seems difficult. Also, there is no well-behaved trace preserving map $A_{n,k}\to A_{n',k'}\otimes A_{n'',k''}$

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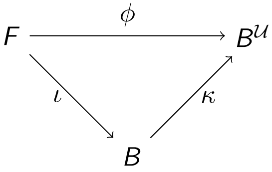
A separable C^* algebra A has quantifier elimination if, for all separable $B \equiv A$, all F finitely generated C^*-algebras, $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ and embeddings $\phi \colon F \to B^{\mathcal{U}}$, $\iota \colon F \to B$ there is $\kappa \colon B \to B^{\mathcal{U}}$ such that the following commutes:



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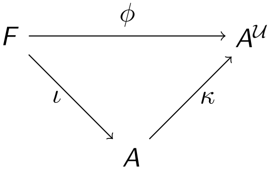

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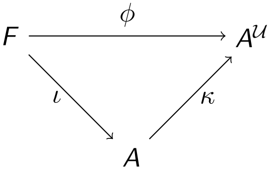

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Theorem (Eagle-Farah-Kirchberg-V., Eagle-Goldbring-V.)

- *In the language of unital* C^**-algebras, the only* C^**-algebras with QE are* $\mathbb{C}$, $\mathbb{C}^2$, $M_2(\mathbb{C})$ *and* $C(2^{\mathbb{N}})$.
- *The only nonabelian* C^**-algebra with* $(\star)$ *is* $M_2(\mathbb{C})$.

One is tempted, therefore, to add a predicate such as the trace. If one considers the language of tracial unital C^*-algebras, then $\mathcal{Z}$ has property $(\star)$ if one restricts itself to those F of the form $F = \bigotimes_{i \leq n} Z_{p_i,q_i}$, where p_i and q_i are coprime for all i.

If A is a Fraïssé limit for the Fraïssé class $\mathcal{K}$, for all the examples we have, A satisfies $(\star)$ whenever $F \in \mathcal{K}$. This is where the absence of HP kicks in.

Question

Does $\mathcal{Z}$ satisfy $(\star)$ in the language of tracial unital C^*-algebras? Does it have QE?

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Thank you!