Forcing and selection principles

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- Selection principles (combinatorial covering properties)
- Combinatorial versions of separability
- Preservation by products

Menger and Hurewicz spaces

X is compact if for every open cover $\mathcal{U}$ of X there exists a finite $\mathcal{V} \subset \mathcal{U}$ such that $X = \bigcup \mathcal{V}$.

X is σ-compact if $X = \bigcup_{n \in \omega} K_n$ for some compact K_n.

Can we reformulate the σ-compactness as a covering property?
1st attempt:
For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

2nd attempt:
for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and each $x \in X$ belongs to $\cup \mathcal{V}_n$ for all but finitely many n.

1 - Menger property
2- Hurewicz property

$$\sigma\text{-compact} \to \text{Hurewicz} \to \text{Menger} \to \text{Lindelöf}.$$

Example: ω^ω is not Menger. Witness:
$\mathcal{U}_n = \big\{\{x : x(n) = k\} : k \in \omega\big\}$.

Basic facts and examples

For analytic sets of reals Menger is equivalent to σ-compact.
In L there exists a co-analytic Menger subspace of ω^ω which is not σ-compact.

Given $x, y \in \omega^\omega$, $x \leq^* y$ means $\{n : x(n) \leq y(n)\}$ is cofinite.

Theorem (Hurewicz 1925)

A zero-dimensional Lindelöf space X is Hurewicz iff $f[X]$ is bounded with respect to $\leq^*$ for any continuous $f : X \to \omega^\omega$.

A zero-dimensional Lindelöf space X is Menger iff $f[X]$ is non-dominating with respect to $\leq^*$ for any continuous $f : X \to \omega^\omega$.

$\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ω^ω. $\mathfrak{d}$ is the minimal cardinality of a dominating subset of ω^ω.

$|X| < \mathfrak{b} \to X$ is Hurewicz.
$|X| < \mathfrak{d} \to X$ is Menger.

Examples of Menger and Hurewicz spaces under CH.

$X \subset \omega^\omega$ is a *Luzin* set if $|X| = \omega_1$ and $|X \cap M| \leq \omega$ for any meager M. Every Luzin set is Menger because concentrated.

$X \subset 2^\omega$ is a *Sierpinski* set if $|X| = \omega_1$ and $|X \cap N| \leq \omega$ for any measure 0 set N.

Theorem (Scheepers 1996)

Let P be compact. $X \subset P$ is Hurewicz iff for every G_δ-set $G \supset X$ there exists a σ-compact F such that $X \subset F \subset G$.

Corollary

Luzin sets are Menger but not Hurewicz. Sierpinski sets are Hurewicz.

More generally: $\mathfrak{b}$-Sierpinski sets are Hurewicz and $\mathfrak{d}$-Luzin sets are Menger.

Examples using forcing

Theorem (Essentially A. Dow)

Let (X, τ) be a Lindelöf space. Then X is Menger in $V^{Fn(\mu,2)}$.

Proof. Two steps. 1. X remains Lindelöf. 2. X becomes Menger. □

Theorem (Folklore)

Every set of ground model reals is Hurewicz after adding uncountably many dominating reals.

The formulation above is of course informal.

Distinguishing between these properties in ZFC

Fremlin and Miller 1988:
In ZFC there exists a Menger non-σ-compact set of reals.

Just, Miller, Scheepers, and Szeptycki 1996:
In ZFC there exists a Hurewicz non-σ-compact set of reals.

Bartoszynski and Tsaban 2006:
In ZFC there exists a set of reals whose all finite powers are Hurewicz non-σ-compact.

Chaber and Pol 2002:
In ZFC there exists a set of reals whose all finite powers are Menger and not Hurewicz.

For curiosity:

Telgarsky 1984:
If the second player has a winning strategy in the Menger game on a hereditarily Lindelöf X, then X is σ-compact.

How to get such ZFC - counterexamples?

A set $X \subset \omega^\omega$ is κ-concentrated on a countable Q, if $|X| \geq \kappa$ and $|X \setminus U| < \kappa$ for any open $U \subset \omega^\omega$ containing Q. If $\kappa \leq \mathfrak{d}$, then $X \cup Q$ is Menger.

Fact. There exists a $\mathfrak{d}$-concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\leq^* d_\beta$ for all $\beta \leq \alpha$. Viewed as a subspace of $(\omega+1)^{\uparrow\omega}$, S is $\mathfrak{d}$-concentrated on $Q = \{x \in (\omega+1)^{\uparrow\omega} : x$ is eventually $\omega\}$. $\square$

Fact. There exists a $\mathfrak{b}$-concentrate set.

Proof. Fix an unbounded $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$ such that $b_\beta \leq^* b_\alpha$ for all $\beta \leq \alpha$. B is $\mathfrak{b}$-concentrated on Q. $\square$

Bartoszynski-Shelah 2001: $B \cup Q$ is Hurewicz.
Bartoszynski-Tsaban 2006: All finite powers of $B \cup Q$ are Hurewicz.

"All $\mathfrak{b}$-concentrated sets are Hurewicz" is independent: wrong under $\mathfrak{b} = \mathfrak{d}$, true in the Miller model.

Preservation by products: reduction to subspaces of 2^ω.

In ZFC there are two normal spaces X, Y with a covering property much stronger than the Hurewicz one such that $X \times Y$ is not Lindelöf (Todorcevic 1995).

Theorem (Z. 2005)

Suppose that X_0, X_1 have properties $\mathcal{P}_0, \mathcal{P}_1$, respectively, where $\mathcal{P}_0, \mathcal{P}_1 \in \{$ Menger, Hurewicz, Menger in all finite powers, Hurewicz in all finite powers $\}$, and $X \times Y$ is Lindelöf but not Hurewicz (resp. Menger).

Then there exist $X', Y' \subset 2^\omega$ having properties $\mathcal{P}_0, \mathcal{P}_1$, respectively, and such that $X' \times Y'$ is not Hurewicz (resp. Menger).

A typical non-preservation by products proof

Theorem (Just, Miller, Scheepers, and Szeptycki 1996)

(CH) There are two Sierpinski (hence Hurewicz) sets S_0, S_1 whose product is not Menger.

Proof. Fix a countable dense $Q \subset 2^\omega$ and write $2^\omega \setminus Q = \{x_\alpha : \alpha < \omega_1\}$. In the construction of a Sierpinski set by transfinite induction at each stage α we can pick a point s_α outside of a given measure zero set $Z_\alpha \subset 2^\omega$. 2^ω has a natural structure of a topological group, and the sum of any two measure 1 sets is the whole group. Choose $s^0_\alpha, s^1_\alpha \in 2^\omega \setminus Z_\alpha$ such that $s^0_\alpha + s^1_\alpha = x_\alpha$ and $s^i_\alpha + \{s^{1-i}_\beta : \beta < \alpha\} \cap Q = \emptyset$. Set $S_i = \{s^i_\alpha : \alpha < \omega_1\}$. □

Further non-preservation results

X is a *γ-space* if $C_p(X) = \{f : X \to \mathbb{R} | f$ is continuous $\}$ with the topology inherited from $\mathbb{R}^X$ is FU.

X is *Rothberger* if For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle V_n : n \in \omega \rangle$ such that $V_n \in \mathcal{U}_n$ and $\bigcup \{V_n : n \in \omega\} = X$.

Each γ-sets is Rothberger and Hurewicz, and all finite powers of a γ-set are γ-sets.

Theorem (Scheepers 1999)

(CH) There exist $X, Y \subset 2^\omega$ such that all finite powers of X, Y are Rothberger but $X \times Y$ is not Menger.

Theorem (A. Miller-Tsaban-Zdomskyy 2016)

(CH) There exist γ-sets $X, Y \subset 2^\omega$ such that $X \times Y$ is not Menger.

Theorem (Barman-Dow 2012)

PFA implies that any finite product of γ-sets of reals is Menger.

Further non-preservation results

Corollary (A. Miller-Tsaban-Zdomskyy 2016)

After adding uncountably many Cohen reals to a model of CH, there exist γ-sets $X, Y \subset 2^\omega$ such that $X \times Y$ is not Hurewicz but all finite powers thereof are Menger.

Theorem (Repovš-Zdomskyy 2010)

$(\mathfrak{b} = \mathfrak{d})$. *There exist $X, Y \subset 2^\omega$ such that all finite powers of X, Y are Menger but $X \times Y$ is not Menger.*

Combinatorial versions of separability: switching points and open sets

Scheepers 1999:
X is *M-separable*, if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, one can pick finite subsets $F_n \subset D_n$ so that $\bigcup_{n\in\omega} F_n$ is dense.

Bella, Bonanzinga, Matveev 2009:
X is *H-separable*, if it satisfies a stronger version of the property above, namely that $\bigcup_{n\in I} F_n$ is dense for all $I \in [\omega]^\omega$ (equivalently, every nonempty open set $O \subset X$ meets all but finitely many F_n's).

Example:
second-countable spaces (even spaces with a countable π-base) are H-separable, and each H-separable space is M-separable.

Countable FU spaces are M-separable.
There seems to be no (consistent) example of a FU and non-H-separable space known.

Connection via the C_p-duality

Theorem (Scheepers 1999)

For a metrizable space X, $C_p(X)$ is M-separable if and only if all finite powers of X are Menger.

Theorem (Bella-Bonanzinga-Matveev 2009)

For a metrizable space X, $C_p(X)$ is H-separable if and only if all finite powers of X are Hurewicz.

As a result, there are countable regular spaces (even countable topological groups) distinguishing between having countable π-base, H-separability, and M-separability.

Preservation by products

$C_p(X \sqcup Y) = C_p(X) \times C_p(Y)$.

If $X \times Y$ doesn't have some property $\mathcal{P}$ inherited by closed subsets, then $(X \sqcup Y)^2$ also doesn't have $\mathcal{P}$.

Now suppose that X^n and Y^n are Menger for all $n \in \omega$ and $X \times Y$ is not Menger. Then $C_p(X), C_p(Y)$ are M-separable, but $C_p(X) \times C_p(Y)$ is not.

Similarly for Hurewicz vs. H-separability.

Now we can just translate. E.g.,

Theorem (A. Miller-Tsaban-Z 2016)

(CH). There are two FU spaces of the form $C_p(*)$ (and hence they are H-separable) whose product is not M-separable.

There are also direct constructions due to Barman, Dow, M. Sakai, and Gruenhage.

Preservation by products: positive results

Theorem (Repovs-Z. 2016)

In the Laver model for the consistency of the Borel's conjecture, the product of any two Hurewicz subspaces of 2^ω has the Menger property.

Note: The conclusion doesn't follow from the Borel's Conjecture.

Theorem (Z. 2016)

In the Miller model the product of any two Menger subspaces of 2^ω has the Menger property.

We don't know whether $\mathfrak{u} < \mathfrak{g}$ suffices.

Preservation by products: positive results

Theorem (Repovs-Z. 2016)

In the Laver model for the consistency of the Borel's conjecture, the product of any two H-separable spaces is M-separable, provided that all of its dense subsets are separable.

Theorem (Z. 2017)

In the Miller model the product of any two M-separable spaces is M-separable, provided that all of its dense subsets are separable.

Weak concentration

Definition

X is *weakly concentrated* if for every collection $\mathsf{Q} \subset [X]^\omega$ which is cofinal with respect to inclusion,

and for every function $R : \mathsf{Q} \to \mathcal{P}(X)$ assigning to each $Q \in \mathsf{Q}$ a G_δ-set $R(Q)$ containing Q,

there exists $\mathsf{Q}_1 \in [\mathsf{Q}]^{\omega_1}$ such that $X \subset \bigcup_{Q \in \mathsf{Q}_1} R(Q)$.

Under CH any subset of 2^ω is weakly concentrated. So the notion might be interesting only under $\mathfrak{c} > \omega_1$.

Lemma

- *In the Laver model every Hurewicz subspace of $\mathcal{P}(\omega)$ is weakly concentrated.*
- *If $\mathfrak{b} > \omega_1$, then a product of a weakly concentrated $X \subset 2^\omega$ and a Hurewicz $Y \subset 2^\omega$ is Menger.*

How concentration works in products

Time permitting, it should be explained on the blackboard why Hurewicz x concentrated is Menger.

The last slide

Thank you for your attention.