SIDE CONDITIONS AND ITERATION THEOREMS

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These notes are based upon lectures given by Itay Neeman at the April 9th, 2016 Appalachian Set Theory workshop. Thomas Gilton was the official note-taker. In these notes, we will develop the machinery necessary to state, prove, and understand a side conditions iteration theorem of Neeman's. A more involved version of the iteration theorem presented here appears in Neeman's forthcoming work which generalizes Baumgartner's result that it's consistent that any $\aleph_1$-dense sets of reals are order-isomorphic.

Before giving an outline of these notes, we briefly survey past results in order to better situate Neeman's work. By "iteration theorem" we mean a statement of the following form: posets satisfying property P can be iterated in such a manner that the whole iteration has property Q. One of the first results of this form is the result of Solovay and Tennenbaum ([11]) that a finite support iteration of c.c.c. posets is still c.c.c.; another example is Shelah's result ([10]) that a countable support iteration of proper forcings is still proper.

By a "side condition" we mean the elementary substructures of a given transitive structure which appear as part of a condition in some forcing. Forcing with side conditions has its origin in the work of Todorčević; see, for instance, ([12]). The motivation for including models in forcing conditions is to make it easier to prove properness: by demanding that certain models appear in conditions, constraints can be put on the so-called "working part" by demanding that it interacts in some apt way with the models.[1] Consequently, it is easier to show that conditions in the forcing are master conditions for many models.

Since the groundbreaking work of Todorčević, there have been many other applications of forcing with side conditions. For instance, Koszmider ([5]) has shown how to force a sequence $\langle f_\alpha : \alpha < \omega_2 \rangle$ of functions $f_\alpha : \omega_1 \longrightarrow \omega_1$ which are increasing mod finite, and Friedman ([2]) has shown how to add a club subset of ω_2 without collapsing cardinals. Additionally, Mitchell ([6]) has used an intricate side conditions forcing to solve a major open problem posed by Shelah, showing that it's consistent that the ideal $I[\omega_2]$ is as small as possible, in the sense that it contains no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$.[2]

In these notes we will be focusing on the side conditions technique invented by Neeman ([8],[7]); this technique has the advantage that the side conditions are $\in$-increasing sequences of models, thereby retaining much of the simplicity and elegance of the original approach of Todorčević. However, in order to avoid undesired

[1] In these initial applications, Todorčević was able to show properness, though ω_2 was collapsed, since the models were $\in$-chains of countable elementary submodels of $H(\omega_2)$. Todorčević ([13]) later modified his method to achieve the preservation of ω_1 and ω_2.

[2] Mitchell's forcing, which is related to Friedman's, adds κ^+-many club subsets of a large cardinal κ, collapses κ to be the new ω_2, and preserves ω_1; see [3] for more details.

collapsing of cardinals above ω_1, the sequences include models of two types, both "small" models and larger, transitive models.

One of our goals is to isolate the class of posets which will be used in the iteration theorem. Roughly, this class consists of countably closed posets with particularly nice "residue systems;" this class of posets subsumes strongly proper posets with countably closed quotients. However, we cannot simply iterate these posets and expect to preserve ω_2. Accordingly, we will use the technique of forcing with side conditions in order to secure properness for a sufficiently large class of models; this in turn will allow us to prove that ω_2 is preserved. Neeman's theorem can thus be described as a side conditions iteration theorem.[3]

We will begin with a review of properness and strong properness in the first section. In section 2, we introduce the technical machinery necessary to analyze certain quotients and to understand the definition of a residue system. In the final section, we review Neeman's method of side conditions and tackle the iteration theorem. We assume that the reader has some familiarity with forcing; it would also benefit the reader to have seen proper forcing.

1. Properness and Strong Properness

In this section, we will work with a poset $\mathbb{P}$. Our goals are to define and briefly explicate properness and strong properness and also to discuss a few of the implications of forcing with posets having these and related properties.

1.1. **Master conditions.**

We begin with notions which have their origin in the work of Shelah ([9], [10]).

Definition 1.1. *A condition $p \in \mathbb{P}$ is a* master condition *for a set M if p forces that $\dot{G} \cap M$ meets every dense $D \subseteq \mathbb{P}$ with $D \in M$.*

Observe that we can swap "dense subset of $\mathbb{P}$ in M" with "maximal antichain of $\mathbb{P}$ in M." We will frequently switch between these without further comment.

Exercise

(1) Let $\mathbb{Q}$ be any c.c.c. forcing (for instance, the poset to add κ-many subsets of ω). Let θ be a large enough regular cardinal and $M \prec H(\theta)$ countable with $\mathbb{Q} \in M$. Show that the maximal condition in $\mathbb{Q}$ is a master condition for M.
(2) Let $\mathbb{Q}$ be any countably-closed forcing, and let M be as in (1). Show that any condition $q \in \mathbb{Q} \cap M$ can be extended to a master condition for M.

There are quite a few equivalent formulations of being a master condition, as the following lemma shows.

Lemma 1.2. *Suppose that $p \in \mathbb{P}$ and that $\mathbb{P} \in M$, where $M \prec H(\theta)$ for some large enough θ. Then the following are equivalent:*

(1) *p is a master condition for M;*
(2) *$p \Vdash M[\dot{G}] \cap V = M$;*

[3]For another example of such a theorem, see ([1]) where Asperó and Mota obtain an iteration theorem for finitely proper posets of size $\aleph_1$; they use symmetric systems of models in order to achieve an $\aleph_2$-c.c. iteration.

(3) *for all $p^* \leq p$ and all dense $D \subseteq \mathbb{P}$ in M, there is an $r \in D \cap M$ which is compatible with p^*.*

Proof. The proof is standard. See ([4], Chapter 31). □

We can think of condition (2) above as saying that nothing in M can name an element of V outside of M. The following exercise gives an example of where this fails drastically.

Exercise Let $\mathbb{P}_0$ be the poset which adds a surjection from ω to ω_1 with finite conditions, and fix a countable $M \prec H(\theta)$, for some large enough regular θ, such that $\mathbb{P}_0 \in M$. Show that

$$\Vdash_{\mathbb{P}_0} M[\dot{G}] \cap \omega_1^V = \omega_1^V,$$

and hence that no condition in $\mathbb{P}_0$ is a master condition for M.

We now define what it means for a poset to be proper for a set M, and we'll show how to derive cardinal preservation results from this.

Definition 1.3. $\mathbb{P}$ *is said to be* proper for M *if for every $p \in M \cap \mathbb{P}$, there is a master condition q for M s.t. $q \leq p$.*[4]

The next well-known lemma shows that if $\mathbb{P}$ is proper for enough countable models, then forcing with $\mathbb{P}$ preserves ω_1.

Lemma 1.4. *Suppose that for all large enough regular θ, there are stationarily many $M \in P_{\omega_1}(H(\theta))$ s.t. $\mathbb{P}$ is proper for M. Then forcing with $\mathbb{P}$ preserves ω_1.*

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name s.t.

$$\mathbb{P} \Vdash \dot{f} \text{ is a function from } \omega \text{ to } \omega_1^V,$$

and let $p \in \mathbb{P}$ be given. We find a $q \leq p$ which forces that the range of $\dot{f}$ is bounded in ω_1. Let θ be large enough so that $\dot{f}, \mathbb{P} \in H(\theta)$ and so that there is a stationary set $\mathcal{S} \subseteq P_{\omega_1}(H(\theta))$ such that $\mathbb{P}$ is proper for each $M \in \mathcal{S}$. Since $\mathcal{S}$ is stationary, we can find an $M \in \mathcal{S}$ s.t. $M \prec H(\theta)$ and $\dot{f}, \mathbb{P}, p$ are all in M. Let $q \leq p$ be a master condition for M. We will show that $q \Vdash \text{ran}(\dot{f}) \subseteq M$; since $M \cap \omega_1$ is an ordinal below ω_1, this suffices.

Thus fix $n < \omega$ and $r \leq q$; we find an $r' \leq r$ and an ordinal $\alpha \in M \cap \omega_1$ s.t. $r' \Vdash \dot{f}(n) = \check{\alpha}$. Let D_n be the dense set of conditions in $\mathbb{P}$ which decide the value of $\dot{f}(n)$. Since $n, \dot{f}, \mathbb{P} \in M$, we have that $D_n \in M$ by elementarity. Using Lemma 1.2(3), we can find an $s \in D_n \cap M$ which is compatible with r. Let $r' \leq s, r$, and let α be such that $s \Vdash \dot{f}(n) = \check{\alpha}$. Observe that $\alpha \in M$, being definable from parameters in M. Since $r' \leq s$, we know that $r' \Vdash \dot{f}(n) = \check{\alpha}$. Finally, as r and n were arbitrary, we have $q \Vdash \text{ran}(\dot{f}) \subseteq M$. □

1.2. **Strong master conditions.**

We now strengthen the definition of a master condition to that of a *strong master condition*; this idea was made explicit in the work of Mitchell ([6]).

[4] A poset is said simply to be *proper* if for all large enough regular θ, there is a club of $M \in P_{\omega_1}(H(\theta))$ such that $\mathbb{P}$ is proper for M. We will not need this definition in these notes.

Definition 1.5. *p is a* strong master condition *for a set M if p forces that $\dot{G} \cap M$ is generic for $\mathbb{P} \cap M$ over V.*

This definition is equivalent to the following: for every maximal antichain A of the poset $\mathbb{P} \cap M$, $p \Vdash \dot{G} \cap A \neq \varnothing$; note that we are *not* assuming that A is an element of M here, nor are we assuming that A is a maximal antichain of $\mathbb{P}$. As in the definition of a master condition, we can swap "maximal antichain of $\mathbb{P} \cap M$" and "dense subset of $\mathbb{P} \cap M$."

Exercise Suppose that $M \prec H(\theta)$ for some large enough regular θ with $\mathbb{P} \in M$, and suppose also that p is a strong master condition for M. Show that p is a master condition for M.

In practice, when proving that a condition $p \in \mathbb{P}$ is a strong master condition for a set M, we work with a more combinatorial characterization of this notion. Roughly, the idea is that p is a strong master condition for M iff for any extension $q \leq p$, there is some "reflection" of q, which we call $\bar{q}$, with $\bar{q} \in M$ satisfying the following property: any extension of $\bar{q}$ inside $\mathbb{P} \cap M$ is compatible with q. We will refer to this process as *amalgamation.* Before we make this equivalence precise, we need a definition.

Definition 1.6. *A* (partial) *function $f : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$ is a* strong residue function for $\mathbb{P}$ at M *if for any $s \in \operatorname{dom}(f)$, every $t \leq f(s)$ with $t \in \mathbb{P} \cap M$ is compatible with s.*

Note that the definition of a strong residue function can be satisfied vacuously. The following proposition connects the existence of non-trivial strong residue functions with the existence of strong master conditions.

Proposition 1.7. *The following are equivalent:*

(1) *$p \in \mathbb{P}$ is a strong master condition for M;*
(2) *there is a strong residue function f for $\mathbb{P}$ at M with $\operatorname{dom}(f)$ dense below p;*
(3) *there is a strong residue function f for $\mathbb{P}$ at M with $\operatorname{dom}(f) \supseteq \{p^* \in \mathbb{P} : p^* \leq p\}$.*

Moreover, such an f can be chosen independently of p in the sense that there is a single partial function $f : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$ such that for any strong master condition p for M, f witnesses (3) for p.

Proof. It is clear that (3) $\implies$ (2). For (2) $\implies$ (1), fix a maximal antichain A in $\mathbb{P} \cap M$. To show that p is a strong master condition, it suffices to show that for all $p^* \leq p$ there is an $r \leq p^*$ s.t.

$$r \Vdash A \cap \dot{G} \neq \varnothing.$$

Let $p^* \leq p$ be given. By extending p^* if needed, we may assume that $p^* \in \operatorname{dom}(f)$; then $f(p^*) \in \mathbb{P} \cap M$. By the maximality of A, there is an $s \in A$ which is compatible with $f(p^*)$ in $\mathbb{P} \cap M$. Let $t \in \mathbb{P} \cap M$ be a common extension. Then by the definition of a strong residue function, t and p^* are compatible in $\mathbb{P}$, so we can find r below t and p^*. Then $r \leq t \leq s$ implies that $r \Vdash s \in A \cap \dot{G}$. Since $r \leq p^*$, we're done.

(1) $\implies$ (3). We define the partial function f uniformly. This will give us both the desired implication and the "moreover" part of the lemma. Given $s \in \mathbb{P}$, if there is a condition $r \in \mathbb{P} \cap M$ s.t. every $t \leq r$ in $\mathbb{P} \cap M$ is compatible with s, let $f(s)$ be such an r. Otherwise, we leave $f(s)$ undefined. By the definition of f, it is clear that f is a strong residue function for $\mathbb{P}$ at M. Let p be a strong master condition

for M, and let $p^* \leq p$; we show that $p^* \in \mathrm{dom}(f)$. Note first that p^* is also a strong master condition for M. Suppose for a contradiction that $p^* \notin \mathrm{dom}(f)$. Then for any $r \in \mathbb{P} \cap M$ there is a condition $t \in \mathbb{P} \cap M$ with $t \leq r$ s.t. t is incompatible with p^*. But this is the same as saying that

$$D := \{t \in \mathbb{P} \cap M : t \text{ is incompatible with } p^*\}$$

is dense in $\mathbb{P} \cap M$. Hence $p^* \Vdash \dot{G} \cap D \neq \varnothing$, since p^* is a strong master condition, contradicting the definition of a filter. □

Definition 1.8. *$\mathbb{P}$ is* strongly proper *for a set M if every $p \in \mathbb{P} \cap M$ extends to a strong master condition for M.*

We say $\mathbb{P}$ is strongly proper for a class $\mathcal{S}$ of models *if it is strongly proper for each $M \in \mathcal{S}$.*

Remark If $\mathbb{P}$ is (non-trivial and) strongly proper for a countable model M, then $\mathbb{P}$ adds reals. Indeed, if G is $\mathbb{P}$-generic over V, then by assumption, $G \cap M$ is generic for $\mathbb{P} \cap M$ over V. However, $\mathbb{P} \cap M$ is a non-trivial, countable poset and hence adds a Cohen real. Consequently no countably closed poset is strongly proper for any countable model M.

Example Let $\mathcal{S}$ consist of all countable elementary submodels of $H(\omega_2)$. Let $\mathbb{Q}$ be the poset consisting of finite $\in$-chains of elements of $\mathcal{S}$ with the ordering defined as follows:

$$\langle N_i : i \leq n\rangle \leq_{\mathbb{Q}} \langle M_j : j \leq m\rangle \text{ iff } \{M_j : j \leq m\} \subseteq \{N_i : i \leq n\}.$$

We show that $\mathbb{Q}$ is strongly proper for $\mathcal{S}$. Given a model $M \in \mathcal{S}$ and a condition $p = \langle K_i : i \leq k\rangle$ in M, we show how to extend p to a strong master condition for M. Define $q := p \cup \{M\}$; we'll produce a strong residue function f for $\mathbb{Q}$ at M with $\mathrm{dom}(f) \supseteq \{q^* : q^* \leq q\}$. Given any $r \leq q$, let $f(r) := r \cap M$, and observe that $r \cap M \in M$ since $r \cap M$ is a finite subset of M.

Now fix any $r \leq q$, and let $t \in M \cap \mathbb{Q}$ with $t \leq f(r) := r \cap M$. Then t and r are compatible. Indeed, $t \cup r$ is a condition since it is the finite $\in$-chain which consists of the models of t, followed by the model M, followed by the models of r above M. Observe that since $\mathcal{S}$ is club in $P_{\omega_1}(H(\omega_2))$, forcing with $\mathbb{Q}$ preserves ω_1, by Lemma 1.4.

The reader should check that that $\mathbb{Q}$ adds a V-fast club in ω_1, i.e., a club $C^* \subseteq \omega_1$ such that for any club $C \subseteq \omega_1$ in V, there is some $\alpha < \omega_1$ such that $C^* \setminus \alpha \subseteq C$.

1.3. **Examples with many strong master conditions.**

Our next goal is to show that given some additional assumptions, strong master conditions are plentiful.

Definition 1.9. *M is a* subcompactness structure *if for some cardinal κ, the following conditions are satisfied:*

1. $\kappa \in M$, $|M| < \kappa$, *and* $M \models$ Extensionality;
2. $M \cap H(\kappa)$ *is transitive;*
3. *the transitive collapse of M is $H(\tau)$ for some regular τ.*

For example, if $\pi : V \longrightarrow V^*$ is an $|H(\gamma)|$-supercompactness embedding with critical point κ, where $\gamma > \kappa$, then in V^*, $\pi'' H(\gamma)$ is a subcompactness structure for $\pi(\kappa)$. So you get plenty of subcompactness structures for κ in V by elementarity.

The following lemma shows that subcompactness structures have a nice "approximation" property.

Lemma 1.10. *Let M be a subcompactness structure. Then for every $a \subseteq M$ which is injectible into $u \cap M$ for some $u \in M$, there is an $a^* \in M$ s.t. $a = a^* \cap M$.*

Proof. Let a and u be as in the statement of the lemma, and let $j : M \longrightarrow H(\tau)$ be the transitive collapse embedding. Set

$$\bar{a}^* := j'' a.$$

We first observe that $\bar{a}^* \in H(\tau)$. Indeed, we know that $\bar{a}^*$ is injectible into $u \cap M$ and so is injectible into $j''(u \cap M) = j(u)$. Since $j(u) \in H(\tau)$ and $\bar{a}^* \subseteq H(\tau)$, we get $\bar{a}^* \in H(\tau)$.

Now we let $a^* := j^{-1}(\bar{a}^*)$. Then $a^* \in M$, and

$$j''(a^* \cap M) = j(a^*) = \bar{a}^* = j'' a.$$

Hence $a^* \cap M = a$. □

Lemma 1.11. *Suppose that M is a subcompactness structure, $\mathbb{P} \in M$, and $M \prec H(\lambda)$ for some λ. Then every master condition for M is a strong master condition for M.*

Proof. Fix a master condition p for M and a maximal antichain A in $\mathbb{P} \cap M$; we show $p \Vdash \dot{G} \cap A \neq \varnothing$. By the previous lemma, we can find some $A^* \in M$ s.t. $A^* \cap M = A$. Now by the elementarity of M, A^* is a maximal antichain in $\mathbb{P}$. Thus since p is a master condition and $A^* \in M$,

$$p \Vdash \dot{G} \cap A^* \cap M \neq \varnothing,$$

and so $p \Vdash \dot{G} \cap A \neq \varnothing$. □

The following lemma gives another situation where you have strong properness, but you might only expect properness.

Lemma 1.12. *Fix a regular cardinal κ, and let $\mathbb{P} = \mathrm{Add}(\kappa, \kappa^+)$. Suppose that $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset s.t.* $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $\kappa + 1$ strategically closed. *Then for all large enough regular θ, the composition $\mathbb{P} * \dot{\mathbb{Q}}$ is strongly proper for all $M \prec H(\theta)$ with $\mathbb{P}, \dot{\mathbb{Q}} \in M$, $|M| = \kappa$, $\kappa \in M$, and ${}^{<\kappa}M \subseteq M$.*[5]

Proof. The idea of the proof is to use the new Cohen subsets which are added to guess appropriate extensions in the second coordinate; since conditions in $\mathbb{P}$ are small enough, no condition will be able to prevent our guess from being right every once in a while. We note that we're viewing conditions in $\mathbb{P}$ as (partial) functions $f : \kappa \times \kappa^+ \longrightarrow \kappa$, not into 2.

Fix an M as in the statement of the lemma as well as a condition $\langle p, \dot{q} \rangle \in M$; we will extend $\langle p, \dot{q} \rangle$ to a strong master condition for M. Fix an ordinal $\alpha < \kappa^+$

[5]Note that these last two conditions hold trivially for $\kappa = \omega$.

outside M. Let $\dot{f} : \kappa \longrightarrow \kappa$ be the αth "subset" of κ added by $\mathbb{P}$, i.e., a $\mathbb{P}$-name for the function

$$\xi \mapsto (\bigcup \dot{G})(\xi, \alpha).$$

Let $\varphi : \kappa \longrightarrow M$ be a surjection of κ onto M. By recursion, we construct a sequence $\langle \dot{q}_\xi : \xi < \kappa \rangle$ of $\mathbb{P}$-names for conditions forced to be in $M[\dot{G}] \cap \dot{\mathbb{Q}}$. set $\dot{q}_0 := \dot{q}$. Given $\dot{q}_{2\xi}$ set $\dot{q}_{2\xi+1}$ to be the $\mathbb{P}$-name $\dot{r}$, where it is forced that

$$\dot{r} = \begin{cases} \varphi(\dot{f}(\xi)) & \text{if } \varphi(\dot{f}(\xi)) \leq_{\dot{\mathbb{Q}}} \dot{q}_{2\xi} \\ \\ \dot{q}_{2\xi} & \text{otherwise} \end{cases}$$

By definition, we have that $\dot{q}_{2\xi+1}$ is forced to be an element of $M[\dot{G}]$. Let $\dot{q}_{2\xi+2}$ be a name forced to be a refinement of $\dot{q}_{2\xi+1}$ played according II's winning strategy in $\dot{\mathbb{Q}}$, and observe that since $M[\dot{G}]$ is forced to be $< \kappa$-closed in $V[\dot{G}]$ and elementary relative to II's winning strategy, it is forced that $\dot{q}_{2\xi+2}$ is in $M[\dot{G}]$.[6] At limit stages $\alpha < \kappa$, fix a name $\dot{q}_\alpha$ for a condition played by II's winning strategy which is forced to be a lower bound for the sequence $\langle \dot{q}_\xi : \xi < \alpha \rangle$; again by the closure and elementarity of $M[\dot{G}]$, it is forced that $\dot{q}_\alpha$ is in $M[\dot{G}]$.

Now let $\dot{q}_\kappa$ be a name forced to be a lower bound for $\langle \dot{q}_\alpha : \alpha < \kappa \rangle$. We will argue that $\langle p, \dot{q}_\kappa \rangle$ is a strong master condition for M; for this it suffices to show that for every dense $D \subseteq (\mathbb{P} * \dot{\mathbb{Q}}) \cap M$, every extension $\langle s, \dot{r} \rangle \leq \langle p, \dot{q}_\kappa \rangle$ can be refined to one which forces that $D \cap \dot{G} \neq \varnothing$. Fix $\langle s, \dot{r} \rangle \leq \langle p, \dot{q}_\kappa \rangle$. Since $|\operatorname{dom}(s)| < \kappa$, we can choose $\nu < \kappa$ large enough so that $\langle \nu, \alpha \rangle \notin \operatorname{dom}(s)$. Set $s_0 := s \cap M$, and note that $s_0 \in M$ since $|s_0| < \kappa$ and ${}^{<\kappa}M \subseteq M$. Now we know that $\langle s_0, \dot{q}_{2\nu} \rangle \in (\mathbb{P} * \dot{\mathbb{Q}}) \cap M$, and since D is dense in $(\mathbb{P} * \dot{\mathbb{Q}}) \cap M$, we may find $\langle t, \dot{u} \rangle \in D$ with $\langle t, \dot{u} \rangle \leq \langle s_0, \dot{q}_{2\nu} \rangle$. Fix $\nu' < \kappa$ s.t. $\varphi(\nu') = \dot{u}$.

We claim that $s' := s \cup t \cup \{\langle \nu, \alpha, \nu' \rangle\}$ is a condition in $\mathbb{P}$. Indeed, since $\operatorname{dom}(t) \cap \operatorname{dom}(s) = \operatorname{dom}(s_0)$, s and t are equal on their common domain. Moreover, $\langle \nu, \alpha \rangle \notin (\operatorname{dom}(s) \cup \operatorname{dom}(t))$ by choice of ν and α. Thus s' is a condition in $\mathbb{P}$.

This implies that $\langle s', \dot{r} \rangle$ is a condition in $\mathbb{P} * \dot{\mathbb{Q}}$. We now show that $\langle s', \dot{r} \rangle \leq \langle t, \dot{u} \rangle$: it is clear that $s' \leq_{\mathbb{P}} t$. Moreover, $s' \Vdash \dot{f}(\nu) = \nu'$, and therefore

$$s' \Vdash \varphi(\dot{f}(\nu)) = \varphi(\nu') = \dot{u} \leq_{\dot{\mathbb{Q}}} \dot{q}_{2\nu}.$$

Consequently, $s' \Vdash \dot{q}_{2\nu+1} = \dot{u}$, and so

$$s' \Vdash \dot{r} \leq \dot{q}_\kappa \leq \dot{q}_{2\nu+1} = \dot{u}.$$

This proves that $\langle s', \dot{r} \rangle \leq \langle t, \dot{u} \rangle$. Since $\langle t, \dot{u} \rangle \in D$, we have $\langle s', \dot{r} \rangle \Vdash \dot{G} \cap D \neq \varnothing$, and we're done. □

2. Quotients

In this section, we carry out a more in-depth analysis of forcing in the presence of strong master conditions. Recall that if $p \in \mathbb{P}$ is a strong master condition for a set M, then p forces that $\dot{G} \cap M$ is $\mathbb{P} \cap M$-generic over V. For the remainder of this section, we fix a poset $\mathbb{P}$ and a collection $\mathcal{S}$ of elementary submodels of some large

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[6] Closure is needed even at the successor stages, since a reply by σ needs to take into account the entire history of the play hitherto.

enough $H(\theta)$. In subsections 2.1 and 2.2, we'll make the additional assumption that $\mathbb{P}$ is strongly proper for $\mathcal{S}$.

2.1. Markers and closure for quotients.

Definition 2.1. *A* marker *for $\mathbb{P}$ and $\mathcal{S}$ is a set $\Gamma \subseteq \mathcal{S} \times \mathbb{P}$ of pairs s.t.*

(1) $(\forall M \in \mathcal{S})$ $(\forall q, q^* \in \mathbb{P})$ *if* $\langle M, q\rangle \in \Gamma$ *and* $q^* \leq q$, *then* $\langle M, q^*\rangle \in \Gamma$;
(2) $(\forall M \in \mathcal{S})$ $(\forall p \in \mathbb{P} \cap M)(\exists q \leq p)$ *s.t.* $\langle M, q\rangle \in \Gamma$.

We'll often use the notation $\Gamma(M) = \{q \in \mathbb{P} : \langle M, q\rangle \in \Gamma\}$. Markers should be thought of as selecting particularly nice strong master conditions.

Definition 2.2. *A marker* Γ witnesses that $\mathbb{P}$ is strongly proper (resp. proper) for $\mathcal{S}$ *if for all* $M \in \mathcal{S}$, *every* $q \in \Gamma(M)$ *is a strong master condition* (resp. master condition) *for* M.

For the remainder of subsection 2.1 and also in subsection 2.2, we assume that Γ is a marker which witnesses that $\mathbb{P}$ is strongly proper for $\mathcal{S}$.

Definition 2.3. *Let $M \in \mathcal{S}$, and fix a $\mathbb{P} \cap M$-generic $\bar{G}$. Define $\mathbb{P}/^M_\Gamma \bar{G}$ to be the poset with underlying set*

$$\left\{p \in \Gamma(M) : p \text{ is compatible with all } \bar{s} \in \bar{G}\right\},$$

and with the ordering inherited from $\mathbb{P}$.

The following lemma is standard, and we omit its proof.

Lemma 2.4. *Fix an $M \in \mathcal{S}$.*

(1) *Suppose that G is $\mathbb{P}$-generic over V and $G \cap \Gamma(M) \neq \varnothing$. Define $\bar{G} := G \cap M$, so that $\bar{G}$ is $\mathbb{P} \cap M$-generic over V. Then $G \cap \Gamma(M)$ is $\mathbb{P}/^M_\Gamma \bar{G}$-generic over $V[\bar{G}]$, and G is the upwards closure in $\mathbb{P}$ of $G \cap \Gamma(M)$. In particular,*
$$V[\bar{G}][G \cap \Gamma(M)] = V[G].$$
(2) *If $\bar{G}$ is generic for $\mathbb{P} \cap M$ over V and H is generic for $\mathbb{P}/^M_\Gamma \bar{G}$ over $V[\bar{G}]$, and if G is defined to be the upwards closure of H in $\mathbb{P}$, then G is generic for $\mathbb{P}$ over V and*
$$V[G] = V[\bar{G}][H].$$

We now start exploring when these quotients are countably closed; the following item gives the definition.

Definition 2.5. *We say that* the quotient of $\mathbb{P}$ to M by Γ is countably closed (resp. strategically countably closed) *if $\mathbb{P} \cap M$ forces that $\mathbb{P}/^M_\Gamma \dot{G}_{\mathbb{P}\cap M}$ is countably closed* (resp. strategically countably closed).

The next lemma shows that this definition is equivalent to a seemingly weaker condition, obtained by switching quantifiers, if $\mathbb{P}$ is countably distributive.

Lemma 2.6. *Suppose that $\mathbb{P}$ is countably distributive. Then for any $M \in \mathcal{S}$ and any $\mathbb{P} \cap M$-generic $\bar{G}$, the following two conditions are equivalent:*

(1) *$\mathbb{P}/^M_\Gamma \bar{G}$ is countably closed in $V[\bar{G}]$;*
(2) *for any sequence $\langle p_n : n \in \omega\rangle$ of elements of $\mathbb{P}/^M_\Gamma \bar{G}$ in $V[\bar{G}]$ and each $\bar{s} \in \bar{G}$, there is a $\mathbb{P}$-lower bound for $\langle p_n : n \in \omega\rangle$ which is compatible with $\bar{s}$.*

Proof. Fix $M \in \mathcal{S}$ and a generic $\bar{G}$ for $\mathbb{P} \cap M$.

(1) implies (2) trivially: $\mathbb{P}/^M_\Gamma \bar{G}$ is countably closed iff for any descending sequence of elements of $\mathbb{P}/^M_\Gamma \bar{G}$ in $V[\bar{G}]$, there is a lower bound in $\mathbb{P}$ which is compatible with *all* elements of $\bar{G}$.

We prove that (2) implies (1) by contradiction. Suppose, then, that (2) holds, but that there is a sequence $\langle p_n : n \in \omega \rangle$ of elements of $\mathbb{P}/^M_\Gamma \bar{G}$ in $V[\bar{G}]$ such that for any lower bound p^* in $\mathbb{P}$, there is an $s \in \bar{G}$ s.t. p^* and s are incompatible in $\mathbb{P}$.[7] Now since $\mathbb{P}$ is countably distributive, $\langle p_n : n \in \omega \rangle \in V$. Thus there is a $t \in \bar{G}$ s.t.

$$(*) \quad t \Vdash_{\mathbb{P} \cap M} (\forall\ \mathbb{P}\text{-lower bounds } p^* \text{ for } \langle \check{p}_n : n \in \omega \rangle)(\exists s \in \dot{G}_{\mathbb{P} \cap M})\ [p^* \perp_{\mathbb{P}} s].$$

Since (2) holds and $t \in \bar{G}$, we can find a lower bound p^* in $\mathbb{P}$ for $\langle p_n : n \in \omega \rangle$ which is compatible with t. Let $q \leq_{\mathbb{P}} t, p^*$, and let G' be $\mathbb{P}$-generic over V with $q \in G'$. Now $q \leq p^* \leq p_0$, and since $p_0 \in \Gamma(M)$, we have $q \in \Gamma(M)$, by definition of a marker. Since Γ witnesses that $\mathbb{P}$ is strongly proper for $\mathcal{S}$, we conclude that q is a strong master condition for M. Hence

$$\bar{G}' := G' \cap M$$

is generic for $\mathbb{P} \cap M$ over V. Now since $t \in \bar{G}'$, because $q \leq t$, and since $(*)$ holds, we know that there is an $s \in \bar{G}'$ s.t. $p^* \perp_{\mathbb{P}} s$. However $p^* \in G'$ and $s \in \bar{G}' \subseteq G'$, contradicting the definition of a filter. □

The purpose of the following lemma is to show that many posets have countable closure for quotients.

Lemma 2.7. *Let $\mathbb{P}$ be countably closed. Suppose that the following hold:*

1. *every two compatible conditions in $\mathbb{P}$ have a greatest lower bound;*
2. *every countable descending sequence of conditions in $\mathbb{P}$ has a greatest lower bound.*

Then for any $M \in \mathcal{S}$, the quotient of $\mathbb{P}$ to M by Γ is countably closed.

Proof. Let $M \in \mathcal{S}$. Fix a $\mathbb{P} \cap M$-generic $\bar{G}$ and a descending sequence $\langle p_n : n \in \omega \rangle$ in $V[\bar{G}]$ of elements of $\mathbb{P}/^M_\Gamma \bar{G}$, noting that this sequence is in V since $\mathbb{P}$ is countably closed. Let p^* be a greatest lower bound for $\langle p_n : n \in \omega \rangle$ in $\mathbb{P}$; we will show that $p^* \in \mathbb{P}/^M_\Gamma \bar{G}$. Since $p^* \in \Gamma(M)$, we just need to show that p^* is compatible with every $t \in \bar{G}$.

Fix a condition $t \in \bar{G}$. For each $n \in \omega$, t and p_n are compatible, so let q_n be a greatest lower bound. We argue that $\langle q_n : n \in \omega \rangle$ is decreasing. Note that for each $n \in \omega$, $q_{n+1} \leq t$, by definition, and also that

$$q_{n+1} \leq p_{n+1} \leq p_n.$$

Thus q_{n+1} is a lower bound for p_n and t, and since q_n is the greatest such, we know that $q_{n+1} \leq q_n$.

Now we can let q^* be a greatest lower bound for $\langle q_n : n \in \omega \rangle$. Then $q^* \leq t$, but also $q^* \leq q_n \leq p_n$ for all n. Since p^* is a greatest lower bound for $\langle p_n : n \in \omega \rangle$, we have $q^* \leq p^*$. Hence q^* witnesses that p^* and t are compatible. □

[7] We're implicitly using the fact that any lower bound p^* is in $\Gamma(M)$ to conclude that if p^* is not in the quotient, then it must be incompatible with some element of $\bar{G}$; $p^* \in \Gamma(M)$ by the downwards closure of markers.

Condition (1) above can be thought of as saying that there are weakest witnesses to compatibility. Both conditions (1) and (2) above are natural in the case when $\leq_{\mathbb{P}}$ is reverse inclusion, since we can simply take unions. The following lemma says that condition (2) is implied by condition (1) at the cost of passing to a forcing isomorphic poset, and hence condition (1) is doing most of the work.

Lemma 2.8. *Suppose $\mathbb{P}$ is countably closed and any two compatible conditions in $\mathbb{P}$ have a greatest lower bound. Then there is a poset $\mathbb{P}^*$ and marker Γ^* for $\mathbb{P}^*$ s.t.*

1. *Γ^* witnesses that $\mathbb{P}^*$ is strongly proper for $\mathcal{S}$;*
2. *$\mathbb{P}^*$ satisfies (1) and (2) of Lemma 2.7;*
3. *$\mathbb{P}$ is isomorphic to a dense subset of $\mathbb{P}^*$.*[8]

Proof. (Sketch) Let $\mathbb{P}^*$ be the poset of (not necessarily strictly) descending sequences $\langle p_n : n \in \omega \rangle$ from $\mathbb{P}$ with the ordering

$$\langle p_n^* : n \in \omega \rangle \leq_{\mathbb{P}^*} \langle p_n : n \in \omega \rangle \text{ iff } (\forall n)(\exists m)\, p_m^* \leq_{\mathbb{P}} p_n.$$

The set of constant sequences is dense in $\mathbb{P}^*$ and is certainly isomorphic to $\mathbb{P}$. It is easy to check that $\mathbb{P}^*$ satisfies both (1) and (2) of Lemma 2.7: for (1), just take greatest lower bounds coordinatewise, and for condition (2), diagonalize the given sequences.

Let Γ^* be the marker defined by $\Gamma^*(M) := \{s \in \mathbb{P}^* : (\exists p \in \Gamma(M))\; s \leq_{\mathbb{P}^*} (n \mapsto p)\}$, where $n \mapsto p$ is the constant function with value p. It is clear that Γ^* is a marker witnessing that $\mathbb{P}^*$ is strongly proper for $\mathcal{S}$, using the properties of Γ and the density of $\mathbb{P}$ in $\mathbb{P}^*$. □

2.2. **Strategic closure.**

Let us provide some motivation for the technicalities in this subsection. In the iteration theorem that we'll eventually prove, many assumptions are made on the class of posets which we want to iterate. However, the posets which actually arise in practice often do not themselves directly satisfy these hypotheses, but are forcing isomorphic to posets which do satisfy them. We must be cautious here, though, since in passing to a forcing isomorphic poset, we might loose some of the properties of our original poset, for instance, countable closure might drop to strategic countable closure.[9] In what follows, we return to some of the ideas already introduced in subsections 1.2 and 2.1, such as residue functions, markers, and quotients, but we study what happens when notions such as strategic closure come into play.

Definition 2.9. $\mathbb{P}$ *has* greatest lower bounds for strategically descending sequences *if the Good player (player II) has a winning strategy in the following game: I begins the game, and I and II alternate playing a descending sequence*

I	p_1		p_3		...	
II		p_2		p_4		...

II wins if there is a greatest lower bound.

Definition 2.10. *A partial function $f : \mathbb{P} \rightharpoonup \mathbb{P}$ is* exact *if it satisfies the following conditions:*

[8]In particular, $\mathbb{P}$ and $\mathbb{P}^*$ produce the same generic extensions.

[9]Strategic countable closure will be preserved, though.

(1) (projection) $(\forall p \in \text{dom}(f))\ \ p \leq f(p)$;
(2) (order preservation) $(\forall p, q \in \text{dom}(f))\ q \leq p \longrightarrow f(q) \leq f(p)$;
(3) (strategic continuity) *player II has a winning strategy in the game where I and II alternate playing a descending sequence of conditions in* $\text{dom}(f)$ *and where II wins if* $\langle p_n : n \geq 1\rangle$ *has a greatest lower bound* p^* *s.t.* $p^* \in \text{dom}(f)$ *and* $f(p^*)$ *is a greatest lower bound for* $\langle f(p_n) : n \geq 1\rangle$.

The strengthening of (3) that requires the payoff conditions for all descending sequences $\langle p_n : n \in \omega\rangle$ in $\text{dom}(f)$, not just ones played strategically, is called *(countable) continuity.*

We now show how to weaken some of the requirements in the previous definition if $\mathbb{P}$ satisfies additional hypotheses.

Lemma 2.11. *Suppose that* $\mathbb{P}$ *is separative. Then for all sufficiently elementary* M, *if* $f : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$ *is a strong residue function for* M, *then condition (2) of Definition 2.10 follows from condition (1).*

Proof. Fix $q, p \in \text{dom}(f)$ with $q \leq p$. If $f(q) \not\leq f(p)$, then since $\mathbb{P}$ is separative, we can find $t \leq f(q)$ s.t. t is incompatible with $f(p)$. As $f(p), f(q) \in M$, we can apply the elementarity of M to find such a $t \in M$. However, f is a strong residue function, and therefore t, q are compatible in $\mathbb{P}$. But $q \leq p \leq f(p)$, so t is compatible with $f(p)$, a contradiction. □

It is not in general true that $p \leq f(p)$ for all residue functions; indeed, under certain conditions, if $p \leq f(p)$, then $f(p)$ is the weakest condition with the "strong residue property," as the following Exercise shows:

<u>Exercise</u> Suppose that $\mathbb{P}$ is a separative poset and that $f : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$ is a strong residue function. If $p \leq f(p)$ for all $p \in \text{dom}(f)$, then for all $p \in \text{dom}(f)$, $f(p)$ is the weakest condition every $\mathbb{P} \cap M$-extension of which is compatible with p.

Now we consider a lemma that shows how to get strategic countable closure of quotients from exact, strong residue functions. It's also true that, under some additional assumptions about the poset, the converse holds. We will comment further on this fact after the lemma.

Lemma 2.12. *Suppose that* $\mathbb{P}$ *is countably distributive and separative. Suppose also that for all* $M \in \mathcal{S}$, ${}^{\omega}M \subseteq M$ *and that* M *is elementary in* $H(\theta)$ *with* $\mathbb{P}$ *and* $\leq_{\mathbb{P}}$ *as additional predicates. For each* $M \in \mathcal{S}$, *if there is an exact, strong residue function* f *for* $\mathbb{P}$ *at* M *with* $\text{dom}(f)$ *dense in* $\Gamma(M)$, *then the quotient of* $\mathbb{P}$ *to* M *by* Γ *is strategically countably closed.*

Proof. Fix a strategy ρ witnessing the strategic continuity of f, and let $\bar{G}$ be generic for $\mathbb{P} \cap M$. We have to show that $\mathbb{P}/^M_\Gamma \bar{G}$ is strategically countably closed. We'll show something stronger, namely, that $\mathbb{P}/^M_\Gamma \bar{G}$ has greatest lower bounds for strategically descending countable sequences. We will use ρ to create a strategy for player II in the quotient $\mathbb{P}/^M_\Gamma \bar{G}$; we first need a claim that will allow us to construct this strategy inductively.

<u>Claim</u>: Suppose that $\langle p'_1, ..., p'_{2n}\rangle$ is a sequence of conditions in $\mathbb{P}$ played according to ρ and that $p'_{2n} \in \mathbb{P}/^M_\Gamma \bar{G}$. Then for every $u \leq p'_{2n}$ in $\mathbb{P}/^M_\Gamma \bar{G}$, there are $p'_{2n+1}, p'_{2n+2} \in \mathbb{P}$ s.t. $\langle p'_1, ..., p'_{2n+2}\rangle$ is a play by ρ, $p'_{2n+2} \leq u$, and $p'_{2n+2} \in \mathbb{P}/^M_\Gamma \bar{G}$.

Proof. (of Claim) Fix $u \in \mathbb{P}/^M_\Gamma \bar{G}$ below p'_{2n}, and suppose that the claim fails. Fix $t \in \bar{G}$ forcing this. That is, t forces in $\mathbb{P} \cap M$ that for any p'_{2n+1}, p'_{2n+2} with $\langle p'_1, ..., p'_{2n+2} \rangle$ played according to ρ, if $p'_{2n+2} \leq u$, then p'_{2n+2} is not in the quotient, i.e., (since $p'_{2n+2} \in \Gamma(M)$) that p'_{2n+2} is incompatible with some element of $\dot{G}_{\mathbb{P}\cap M}$.

Since t, u are compatible in $\mathbb{P}$, let $r \leq t, u$. Note that $r \leq u \in \Gamma(M)$. Now by the density of $\mathrm{dom}(f)$ in $\Gamma(M)$, there is $p \leq r$ with $p \in \mathrm{dom}(f)$; moreover,

$$p \leq r \leq u \leq p'_{2n},$$

so p is a legal move for I after $\langle p'_1, ..., p'_{2n} \rangle$. Set $p'_{2n+1} := p$, and let p'_{2n+2} be the reply by ρ. Note that $p'_{2n+2} \leq r \leq t$.

We will now derive a contradiction: since $p'_{2n+2} \in \Gamma(M)$, it is a strong master condition for M. Let G' be $\mathbb{P}$-generic containing p'_{2n+2}, and define $\bar{G}' := G' \cap M$, noting that $\bar{G}'$ is $\mathbb{P} \cap M$-generic over V. Now p'_{2n+2} is compatible with all elements of $\bar{G}'$, and hence $p'_{2n+2} \in \mathbb{P}/^M_\Gamma \bar{G}'$. However, $t \in \bar{G}'$ also, contradicting the fact that t forces p'_{2n+2} out of the quotient. □

To complete the proof of the lemma, we work in $V[\bar{G}]$ and describe a winning strategy for II. Suppose that a play $\langle p_1, ..., p_{2n} \rangle$ in the quotient has been determined. We assume inductively that we have auxiliary conditions $p'_1, ..., p'_{2n}$ in $\mathbb{P}$ s.t. $\langle p'_1, ..., p'_{2n} \rangle$ is a play by ρ and $p'_{2n} = p_{2n}$. Let I play $p_{2n+1} \leq p_{2n} = p'_{2n}$.

By the last claim (with $u := p_{2n+1}$), there are conditions p'_{2n+1}, p'_{2n+2} in $\mathbb{P}$ s.t.

(i) $p'_{2n+2} \leq p_{2n+1}$;
(ii) $p'_{2n+2} \in \mathbb{P}/^M_\Gamma \bar{G}$;
(iii) $\langle p'_1, ..., p'_{2n+2} \rangle$ is a play by ρ.

Let II respond with $p_{2n+2} := p'_{2n+2}$.

After ω-many steps, we have to show that $\langle p_{2n} : n \geq 1 \rangle = \langle p'_{2n} : n \geq 1 \rangle$ has a greatest lower bound in $\mathbb{P}/^M_\Gamma \bar{G}$. First note that this sequence is in V by the countable distributivity of $\mathbb{P}$. Since ρ witnesses the strategic continuity of f, $\langle p'_{2n} : n \geq 1 \rangle$ has a greatest lower bound p^* such that $p^* \in \mathrm{dom}(f)$ and $f(p^*)$ is a greatest lower bound for $\langle f(p_{2n}) : n \geq 1 \rangle$. To finish the proof, we just need to show that $p^* \in \mathbb{P}/^M_\Gamma \bar{G}$.

Since $p^* \in \Gamma(M)$, we must show that p^* is compatible with all $s \in \bar{G}$. For this, since f is a strong residue function, it is enough to show that $f(p^*) \in \bar{G}$. Fix a condition $t \in \bar{G}$ which forces that every element of $\langle p'_{2n} : n \geq 1 \rangle$ is compatible with all elements of $\dot{G}_{\mathbb{P}\cap M}$.[10] Then for each n, every extension of t in $\mathbb{P} \cap M$ is compatible with all p'_{2n}, i.e., t has the "strong residue property" w.r.t. each p'_{2n}. By the exercise after Lemma 2.11, it follows that

$$t \leq f(p'_{2n}), \quad \text{for all } n \in \omega,$$

i.e., that t is a lower bound for the sequence $\langle f(p'_{2n}) : n \geq 1 \rangle$. But $f(p^*)$ is a greatest lower bound for this sequence. Hence $t \leq f(p^*)$, and so $f(p^*) \in \bar{G}$. □

The following lemma states a converse to the previous lemma; we will not provide its proof. We'd like to remind the reader that we're assuming that $\mathbb{P}$ is strongly proper for $\mathcal{S}$ as witnessed by the marker Γ.

[10] We're implicitly using that $\langle p'_{2n} : n \in \omega \rangle \in V$ here.

Lemma 2.13. *Suppose that $\mathbb{P}$ is a separative poset which has greatest lower bounds for strategically descending countable sequences, and let σ be a strategy for $\mathbb{P}$. Suppose further that for all $M \in \mathcal{S}$, ${}^{\omega}M \subseteq M$, $M \prec (H(\theta), \in, \mathbb{P}, \leq_{\mathbb{P}}, \sigma)$, and the quotient of $\mathbb{P}$ to M by Γ is strategically countably closed. Then for each $M \in \mathcal{S}$, there is an exact, strong residue function f_M for $\mathbb{P}$ at M with $\mathrm{dom}(f_M)$ dense in $\Gamma(M)$.*

Thus combining Lemmas 2.12 and 2.13, we see that, under additional assumptions about $\mathbb{P}$ and the models $M \in \mathcal{S}$, the existence of an exact strong residue function for $M \in \mathcal{S}$ is equivalent to the countable closure of the quotient of $\mathbb{P}$ to M by Γ.

2.3. **Residue systems.**

We are now ready to define the main technical apparatus that we'll need for Neeman's iteration theorem, a *residue system*. We first define a strong residue system; after this, we'll weaken the definition to that of a weak residue system, which is what will be used in the theorem.

For the remainder of this section, we drop our assumptions that $\mathbb{P}$ is strongly proper for $\mathcal{S}$ and that Γ witnesses this.

Definition 2.14. *Let Γ be a marker for $\mathbb{P}$ and a class $\mathcal{S}$ of structures. A* residue system *for $\mathbb{P}, \mathcal{S}$, and Γ is a sequence $\langle f_M : M \in \mathcal{S}\rangle$ of functions s.t.*

(1) $(\forall M \in \mathcal{S})\ f_M : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$;
(2) $(\forall M \in \mathcal{S})\ \mathrm{dom}(f_M)$ *is a dense subset of* $\Gamma(M)$.

The sequence $\langle f_M : M \in \mathcal{S}\rangle$ is a strong residue system *if it also satisfies*

(3) $(\forall M \in \mathcal{S})\,(\forall p \in \mathrm{dom}(f_M))$ *every $t \leq f_M(p)$ in $\mathbb{P} \cap M$ is compatible with p in $\mathbb{P}$.*

We also say that a residue system $\langle f_M : M \in \mathcal{S}\rangle$ is exact *if each f_M is exact in the sense of Definition 2.10.*

<u>Remark</u>: By previous work, if Γ witnesses that $\mathbb{P}$ is strongly proper for $\mathcal{S}$, and if $\mathbb{P}$ has strategically countably closed quotients, then by passing to forcing-isomorphic $\mathbb{P}', \Gamma'$ if needed we may assume that there is an exact strong residue system for $\mathbb{P}, \mathcal{S}$, and Γ.

The following Definition is apt to appear strange at first; we will provide motivation after its statement.

Definition 2.15. *A* weak residue system *for $\mathbb{P}, \mathcal{S}$, and Γ is a sequence $\langle f_M : M \in \mathcal{S}\rangle$ satisfying (1) and (2) above and the following weakening of (3):*

(w3) $(\forall M \in \mathcal{S})\,(\forall p \in \mathrm{dom}(f_M))\,(\forall t \in \mathbb{P} \cap M)$ *if there is an $\bar{M} \in \mathcal{S} \cap M$ s.t. $f_{\bar{M}}(t) = f_M(p)$, then t is compatible with p.*

Observe that any exact, strong residue system is also a weak residue system: by exactness, $f_{\bar{M}}(t) = f_M(p)$ implies $t \leq f_M(p)$, so instances of (w3) are a subset of instances of (3).

Let's motivate Definition 2.15: in the definition of a strong residue system, we require, among other things, that each $f_M : \mathbb{P} \rightharpoonup \mathbb{P} \cap M$ is a strong residue function, and therefore, if $p \in \mathrm{dom}(f_M)$, then *any* extension of $f_M(p)$ inside of M is compatible with p. Now for (w3), we only require that p be compatible with those $t \in M \cap \mathbb{P}$ which project in the same way as p, in the sense that for some $\bar{M} \in \mathcal{S} \cap M$,

$$f_{\bar{M}}(t) = f_M(p).$$

We should think of $\bar{M}$ as a reflection of M inside M itself, and this will come up in applications of the elementarity of M. The following lemma illustrates this by showing that for sufficiently elementary M, the existence of a weak residue system implies properness.

Lemma 2.16. *Let $\langle f_M : M \in \mathcal{S}\rangle$ be a weak residue system for $\mathbb{P}, \mathcal{S}$, and Γ. Let U be transitive with a predicate $A \subseteq U$ s.t. all of the objects $\mathbb{P}, \mathcal{S}, \Gamma$, and $\langle f_M : M \in \mathcal{S}\rangle$ are definable over $(U; A)$. Let θ^* be large enough, and let $M^* \prec H(\theta^*)$, with $U, A \in M^*$. Finally, suppose that $M^* \cap U \in \mathcal{S}$. Then $\mathbb{P}$ is proper for M^*.*

Proof. Let $p \in M^*$, and we'll find an extension of p which is a master condition for M^*. Define $M := M^* \cap U$. Note that $\mathbb{P} \subseteq U$, and so $p \in M$. By the definition of a marker, we can find $p^* \in \Gamma(M)$ with $p^* \leq p$. We claim that p^* is a master condition for M.

Suppose, for a contradiction, that p^* is not a master condition for M. Then by extending p^* if necessary, we may assume that there is a specific $D \in M^*$ which is dense open in $\mathbb{P}$ s.t. $p^* \in D$ and which also satisfies

$$p^* \Vdash \dot{G} \cap M \cap D = \varnothing;$$

note that even with these possible modifications, p^* is still an element of $\Gamma(M)$. By definition of a residue system, $\mathrm{dom}(f_M)$ is dense in $\Gamma(M)$, and so we may further assume that $p^* \in \mathrm{dom}(f_M)$. Define $r := f_M(p^*)$.

We now apply the elementarity of M^* to "reflect" M inside of itself. Observe that the following is satisfied in $H(\theta^*)$:

$$(\exists \bar{M} \in \mathcal{S})\,(\exists p' \in \mathbb{P})\,[p' \in D \cap \mathrm{dom}(f_{\bar{M}}) \text{ and } f_{\bar{M}}(p') = r];$$

this is satisfied in $H(\theta^*)$ since M and p^* are witnesses. Also observe that the parameters in this formula, namely $\mathcal{S}$, $\mathbb{P}$, D, $\langle f_M : M \in \mathcal{S}\rangle$, and r, are all elements of M^* (since $U, A \in M^*$). By the elementarity of M^*, we can find a model $\bar{M} \in \mathcal{S} \cap M$ and a condition $p' \in M \cap D \cap \mathrm{dom}(f_{\bar{M}})$ such that

$$f_{\bar{M}}(p') = r = f_M(p^*).$$

Then by (w3), p' is compatible with p^*. This is a contradiction since p^* forces that $\dot{G} \cap M \cap D$ is empty. □

3. Side Conditions Iterations

In this section, we will briefly introduce Neeman's method of forcing with models as side conditions, and then we will develop the machinery to prove the desired iteration theorem. Much of this section is definition-heavy, and we refer the reader to [7] and [8] for more details.

3.1. **Side Conditions.**

The first of our definitions will introduce the models that we will use as side conditions, as well as some notation that we will use for the remainder of the section.

Definition 3.1. *$\mathcal{S}$ and $\mathcal{T}$ are* suitable for side conditions iterations at δ and K *if* [11] *the following conditions are satisfied:*

(1) *K is transitive and countably closed, $\delta \in K$ is a regular cardinal, and* $K \models \mathsf{ZFC} -$ Powerset.
(2) *$\mathcal{T}$ is an $\in$-linearly ordered set of transitive W each of which satisfy*
   (i) $W \prec K$;
   (ii) $W \in K$;
   (iii) *W is countably closed;*
   (iv) *ω_1 and δ are elements of W.*

   Moreover, setting[12]
$$C_{\mathcal{T}} := \{W \cap On : W \in \mathcal{T}\},$$
   $C_{\mathcal{T}}$ is required to be $\geq \delta$ closed below the ordinal $K \cap On$. For each $\alpha \in C_{\mathcal{T}}$, we let $W(\alpha)$ be the unique element, W, of $\mathcal{T}$ s.t. $W \cap On = \alpha$.
(3) *every element of $\mathcal{S}$ is a $< \delta$-sized elementary submodel of K which satisfies the following additional properties:*
   (i) $M \in K$;
   (ii) $\omega_1, \delta \in M$;
   (iii) *M is countably closed;*
   (iv) $M \cap \delta \in On$.

   Moreover, we require that every $M \in \mathcal{S}$ is closed under the partial function $\alpha \mapsto W(\alpha)$ as well as the function $\beta \mapsto C_{\mathcal{T}} \cap \beta$.
(4) *for every $M \in \mathcal{S}$ and every $\gamma \in C_{\mathcal{T}}$, if $M \cap On \subseteq \gamma$, then $M \in W(\gamma)$.*
(5) *if $W \in \mathcal{T}$, $M \in \mathcal{S}$, and $W \in M$, then $M \cap W \in \mathcal{S}$.*[13]

We refer to the elements of $\mathcal{S} \cup \mathcal{T}$ as nodes. *The $M \in \mathcal{S}$ are $\mathcal{S}$mall nodes, and the $W \in \mathcal{T}$ are $\mathcal{T}$ransitive nodes.*

We now show how to construct a poset from suitable $\mathcal{S}$ and $\mathcal{T}$.

Definition 3.2. *Let $\mathcal{S}, \mathcal{T}$ be suitable. Define $\mathbb{P}(\mathcal{S}, \mathcal{T})$ to be the poset consisting of sequences $\langle M_\xi : \xi < \gamma \rangle$ for some $\gamma < \omega_1$ s.t.*

(1) *for each $\xi < \gamma$, $M_\xi \in \mathcal{S} \cup \mathcal{T}$;*
(2) (Cofinal) *for each $\alpha < \gamma$, $\{\xi < \alpha : M_\xi \in M_\alpha\}$ is cofinal in α, and in particular, if $\alpha + 1 < \gamma$, then $M_\alpha \in M_{\alpha+1}$;*
(3) (Closure under intersections) *if $M, W \in \{M_\xi : \xi < \gamma\}$ satisfy that $M \in \mathcal{S}$, $W \in \mathcal{T}$, and $W \in M$, then $M \cap W \in \{M_\xi : \xi < \gamma\}$.*

The ordering is reverse inclusion, i.e.,
$$\langle M^*_\xi : \xi < \gamma^* \rangle \leq_{\mathbb{P}(\mathcal{S},\mathcal{T})} \langle M_\zeta : \zeta < \gamma \rangle \text{ iff } \{M^*_\xi : \xi < \gamma^*\} \supseteq \{M_\zeta : \zeta < \gamma\}.$$

Remark By Claim 2.12 of [7], condition (3) implies full closure under intersections. Additionally, observe that by condition (2), we can recover $\langle M_\xi : \xi < \gamma \rangle$ from $\{M_\xi : \xi < \gamma\}$ by von Neumann rank. Abusing notation, we'll conflate the two.

[11] Think of δ as ω_2 or some cardinal which becomes ω_2 in a generic extension, and think of K as some sufficiently large $H(\tau)$.
[12] Note that $W \cap$ On uniquely determines W since $\mathcal{T}$ is linearly ordered by $\in$.
[13] Observe that by (4), this implies that $M \cap W \in W$.

We now introduce quite a bit of notation that will be useful later in this section. Given an $\alpha \in C_{\mathcal{T}}$, we define

$$\mathcal{S} \restriction \alpha := \mathcal{S} \cap W(\alpha) \text{ and } \mathcal{T} \restriction \alpha := \mathcal{T} \cap W(\alpha).$$

We also use $W(\alpha)^+$ to denote the first transitive node above $W(\alpha)$, if there is one, and to denote K otherwise. Further, if $\alpha \in C_{\mathcal{T}}$, then we let

$$\mathcal{S}^+(\alpha) := \left\{ M \in \mathcal{S} : M \in W(\alpha)^+ \wedge \alpha \in M \right\}.$$

For a condition $s \in \mathbb{P}(\mathcal{S}, \mathcal{T})$, we define

$$s \restriction \alpha := s \cap W(\alpha).$$

Note that $s \restriction \alpha \in \mathbb{P}(\mathcal{S} \restriction \alpha, \mathcal{T} \restriction \alpha)$.

In our analysis of conditions $s \in \mathbb{P}(\mathcal{S}, \mathcal{T})$, we will often use interval notation. For example, we use $[M, W)_s$ to denote all nodes of s from M up to, but not including, W. Certain intervals were isolated in Neeman's study, and they play an important role in showing strong properness. These intervals are called *residue gaps* and arise as follows: given a small node $M \in s$ and a transitive node $W \in M \cap s$, we refer to $[M \cap W, W)_s$ as the *residue gap* of s in M. The following point, though simple, is crucial: the interval $[M \cap W, W)_s$ is contained in W and is thus *disjoint from* M.

Now that we have this notation at our disposal, we begin by defining our residue functions. Many of the details in what follows will be passed over, and the interested reader is invited to peruse [7] and [8].

Definition 3.3. *Given a node $Q \in \mathcal{S} \cup \mathcal{T}$ and a condition $s \in \mathbb{P}(\mathcal{S}, \mathcal{T})$ with $Q \in s$, we define*

$$\operatorname{res}_Q(s) := s \cap Q.$$

We refer to $\operatorname{res}_Q(s)$ *as the* residue of s in Q.

Observe that $\operatorname{res}_Q(s)$ is only defined when $Q \in s$ and that $\operatorname{res}_Q(s) \in Q$, since it is a countable subset of Q, and every member of $\mathcal{S} \cup \mathcal{T}$ is closed under countable sequences. The following lemma gives a description of the residues of conditions.

Lemma 3.4. *If $W \in s$ is a transitive node, then $\operatorname{res}_W(s)$ is the initial segment of s consisting of all the nodes of s below W, and if $M \in s$ is a small node, then $\operatorname{res}_M(s)$ consists precisely of the nodes of s below M, except for the ones in residue gaps of s in M.*

We need an omnibus lemma about $\mathbb{P}(\mathcal{S}, \mathcal{T})$ before we proceed.

Lemma 3.5.

(1) *For each node Q, the partial function $s \mapsto \operatorname{res}_Q(s)$ with domain*

$$\{s \in \mathbb{P}(\mathcal{S}, \mathcal{T}) : Q \in s\}$$

is a strong residue function. Moreover, if $t \in Q$ extends $\operatorname{res}_Q(s)$, then:
   (a) *if $Q \in \mathcal{T}$, then $s \cup t$ is a condition;*
   (b) *if $Q \in \mathcal{S}$, then the closure of $s \cup t$ under intersections is a condition;*

(2) *This strong residue function is exact and satisfies full continuity;*

(3) *For each $M \in \mathcal{S}$,*

$$\Gamma(M) := \{s \in \mathbb{P}(\mathcal{S}, \mathcal{T}) : M \in s\}$$

is a marker for $\mathbb{P}(\mathcal{S}, \mathcal{T})$ and $\mathcal{S}$ which witnesses that $\mathbb{P}(\mathcal{S}, \mathcal{T})$ is strongly proper for $\mathcal{S}$;

(4) *For each $M \in \mathcal{S}$, the quotient of $\mathbb{P}(\mathcal{S},\mathcal{T})$ to M by Γ is countably closed.*

The following lemma will be quite useful for the iteration theorem.

Lemma 3.6. *Given any $s \in \mathbb{P}(\mathcal{S},\mathcal{T})$ and any $W = W(\alpha) \in \mathcal{T}$, there is a condition $s^* \leq s$ with $W(\alpha) \in s^*$; moreover, every small node in s^* is of the form $M \cap R$ for some transitive node R and small node $M \in s$. Hence, given any condition s, the set of nodes*

$$\left\{M \cap W(\alpha)^+ : \alpha \in M \wedge M \in s \cap \mathcal{S}\right\}$$

is $\in$-linearly ordered.[14]

Proof. (Sketch) In Claim 2.5 of [8], Neeman showed that the first part of this lemma is true.[15]

To see that this implies the second part of the lemma, fix a condition s and a transitive node $W(\alpha)$. By the first part of the lemma, we can find a condition $s^* \leq s$ with both $W(\alpha)$ and $W(\alpha)^+$ appearing on the s^*-sequence, if $W^+(\alpha) \neq K$. Argue using the cofinality condition, Definition 3.2(2), that the nodes under consideration must then be linearly ordered. □

3.2. **The Iteration Theorem.**

We now turn our attention towards the iteration theorem. We begin by fixing sequences $\vec{\mathbb{Q}} = \langle \dot{\mathbb{Q}}_\xi : \xi \in \mathrm{dom}(\vec{\mathbb{Q}})\rangle$, $\vec{\Gamma} = \langle \dot{\Gamma}_\xi : \xi \in \mathrm{dom}(\vec{\Gamma})\rangle$ such that $\mathrm{dom}(\vec{\mathbb{Q}}) = \mathrm{dom}(\vec{\Gamma}) \subseteq C_{\mathcal{T}}$. Our goal is to define the *side conditions iteration* of $\vec{\mathbb{Q}}$ and $\vec{\Gamma}$, denoted $\mathbb{A}(\mathcal{S},\mathcal{T},\vec{\mathbb{Q}},\vec{\Gamma})$, or more simply by $\mathbb{A}$. We will do this recursively, making the assumption that for all $\xi \in \mathrm{dom}(\vec{\mathbb{Q}})$,

$$\mathbb{A} \restriction \xi := \mathbb{A}(\mathcal{S} \restriction \xi, \mathcal{T} \restriction \xi, \vec{\mathbb{Q}} \restriction \xi, \vec{\Gamma} \restriction \xi)$$

is the side conditions iteration of $\vec{\mathbb{Q}} \restriction \xi$ and $\vec{\Gamma} \restriction \xi$ using models from $\mathcal{S} \restriction \xi$ and $\mathcal{T} \restriction \xi$. Additionally we assume that for all $\xi \in \mathrm{dom}(\vec{\mathbb{Q}})$, $\dot{\mathbb{Q}}_\xi$ is an $\mathbb{A} \restriction \xi$-name for a poset contained in $W^+(\xi)[\dot{G}_{\mathbb{A}\restriction\xi}]$, and that $\dot{\Gamma}_\xi$ is an $\mathbb{A} \restriction \xi$-name forced to be a marker for $\dot{\mathbb{Q}}_\xi$ and the class

$$\left\{M[\dot{G}_{\mathbb{A}\restriction\xi}] : M \in \mathcal{S}^+(\xi)\right\}$$

of models.

In the definition of $\mathbb{A}$, we need to define first a dense subset, which we call the "inner" poset and which is denoted by $\mathbb{A}^{\mathrm{in}}$; the necessity of doing this is mainly a technicality and will be explained after the definition.

Definition 3.7. *Define $\mathbb{A}^{\mathrm{in}}$ to be the poset consisting of pairs $\langle s, p\rangle$ s.t.*

(1) *$s \in \mathbb{P}(\mathcal{S},\mathcal{T})$;*
(2) *p is a countable, partial function on $\mathrm{dom}(\vec{\mathbb{Q}})$ with*

$$\mathrm{dom}(p) \subseteq \{\alpha : W(\alpha) \in s\};$$

---

[14]Note that these nodes are not necessarily in s, but if they were, the lemma would be completely obvious.

[15]Note, however, that in [8], this is proven a slightly different context, namely, one wherein conditions are *finite* sequences of nodes rather than countable sequences, as in the current context. An easy check, however, shows that [8], Claim 2.5 still goes through with countable length sequences and countably-closed models.

(3) *for any* $\alpha \in \mathrm{dom}(p)$, $p(\alpha)$ *is a canonical* $\mathbb{A} \restriction \alpha$*-name for an element of* $\dot{\mathbb{Q}}_\alpha$;
(4) *for any* $\alpha \in \mathrm{dom}(p)$ *and any small* $M \in s$, *if* $\alpha \in M$, *then*

$$\langle s \restriction \alpha, p \restriction \alpha \rangle \Vdash_{\mathbb{A}\restriction\alpha} p(\alpha) \in \dot{\Gamma}_\alpha(M \cap W(\alpha)^+).^{16}$$

The ordering on $\mathbb{A}^{\mathrm{in}}$ *is the natural one:* $\langle s^*, p^* \rangle \leq \langle s, p \rangle$ *iff*
(a) $s^* \leq s$ *in* $\mathbb{P}(\mathcal{S}, \mathcal{T})$;
(b) *dom*$(p^*) \supseteq \mathrm{dom}(p)$;
(c) *for each* $\alpha \in \mathrm{dom}(p)$,

$$\langle s^* \restriction \alpha, p^* \restriction \alpha \rangle \Vdash_{\mathbb{A}\restriction\alpha} p^*(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha).$$

Note that this ordering makes sense even for pairs $\langle s, p \rangle$ *which just satisfy (1)-(3).*

The poset $\mathbb{A}$ *is defined to be the set of all pairs* $\langle s, p \rangle$ *satisfying (1)-(3) above and the following weakening of* (4) :
(4') *there is a condition* $\langle s', p' \rangle \in \mathbb{A}^{\mathrm{in}}$ *s.t.* $\langle s', p' \rangle \leq \langle s, p \rangle$.

Let us now explain the necessity of defining $\mathbb{A}^{\mathrm{in}}$ and $\mathbb{A}$ as we did. In short, we need to expand $\mathbb{A}^{\mathrm{in}}$ to $\mathbb{A}$ in order to make sure that our reside functions map into the poset. In more detail, we are eventually going to show that, under some additional assumptions, $\mathbb{A}$ is proper for many uncountable models, and we will argue this using residue functions as in Lemma 2.16. However, the natural residue functions that we'd like to define on $\mathbb{A}^{\mathrm{in}}$ do not necessarily map into $\mathbb{A}^{\mathrm{in}}$. By enlarging $\mathbb{A}^{\mathrm{in}}$ slightly, we will avoid this problem; note that since $\mathbb{A}^{\mathrm{in}}$ is dense inside $\mathbb{A}$, they both generate the same generic extensions. One final comment: one can check by induction on the ordinal height of K that $\langle s, p \rangle \in \mathbb{A}^{\mathrm{in}}$ implies that $\langle s, \restriction \alpha, p \restriction \alpha \rangle \in \mathbb{A}^{\mathrm{in}} \restriction \alpha$, and similarly with $\mathbb{A}$. The former fact is used implicitly in Definition 3.7, for example, in conditions (c) and (4).

We now consider a few facts about the poset just defined. The first item on our agenda is to show that for any condition $\langle s, p \rangle \in \mathbb{A}$ and any transitive node $W(\alpha)$, we can always add $W(\alpha)$ to s and α to $\mathrm{dom}(p)$.

Lemma 3.8. *Assume that* $\dot{\mathbb{Q}}_\alpha$ *is forced in* $\mathbb{A} \restriction \alpha$ *to be strategically countably closed and that the poset* $\mathbb{A} \restriction \alpha$ *and the name* $\dot{\mathbb{Q}}_\alpha$ *are definable in* K *from* α. *Further assume that a name forced to be equal to* $\dot{\Gamma}_\alpha(M)$ *is uniformly definable in* K *from* α *and* $M \in \mathcal{S}^+(\alpha)$. *Then for any condition* $\langle s, p \rangle \in \mathbb{A}$ *and any* $\alpha \in \mathrm{dom}(\vec{\mathbb{Q}}) \subseteq C_{\mathcal{T}}$, *there is* $\langle s^*, p^* \rangle \leq \langle s, p \rangle$ *with* $\alpha \in \mathrm{dom}(p^*)$.

Proof. Fix $\alpha \in \mathrm{dom}(\vec{\mathbb{Q}})$ and a condition $\langle s, p \rangle \in \mathbb{A}$; we may as well suppose that $\langle s, p \rangle \in \mathbb{A}^{\mathrm{in}}$. If $\alpha \in \mathrm{dom}(p)$ already, then we are done, so suppose that $\alpha \notin \mathrm{dom}(p)$. We claim that without loss of generality, we may assume that $W(\alpha) \in s$. Indeed, suppose that $W(\alpha) \notin s$. By Lemma 3.6, there is $s' \leq s$ with $W(\alpha) \in s'$, and moreover, every small node of s' is the intersection of a small node of s with a transitive node. Using this last fact, it is easy to check that $\langle s', p \rangle$ satisfies condition (4) of Definition 3.7. Then $\langle s', p \rangle$ is a condition in $\mathbb{A}^{\mathrm{in}}$ extending $\langle s, p \rangle$ such that $W(\alpha) \in s'$.

[16] "$\dot{\Gamma}_\alpha(M \cap W(\alpha)^+)$" should more properly be written "$\dot{\Gamma}_\alpha((M \cap W(\alpha)^+)[\dot{G}_{\mathbb{A}\restriction\alpha}])$." Similar comments apply in our uses of $\dot{\Gamma}_\alpha$ below.

We continue now under this assumption, aiming to find p^* extending p with $\mathrm{dom}(p^*) = \mathrm{dom}(p) \cup \{\alpha\}$ so that $\langle s, p^* \rangle \in \mathbb{A}^{\mathrm{in}}$. By the previous lemma, the nodes

$$\left\{M \cap W(\alpha)^+ : \alpha \in M \wedge M \in s \cap \mathcal{S}\right\}$$

are linearly ordered, and so we can list them in $\in$-increasing order as $\langle N_i : i < \gamma \rangle$ for some $\gamma < \omega_1$; we suppose for simplicity that $\lim(\gamma)$. The idea for the rest of the proof is to simply work our way through each of these models and get inside the relevant markers; the details are as follows.

We define an $\mathbb{A} \restriction \alpha$-name τ, which will then be $p^*(\alpha)$, by showing how to interpret τ in arbitrary $\mathbb{A} \restriction \alpha$-generic extensions. Let G_α be $\mathbb{A} \restriction \alpha$-generic, and work in $V[G_\alpha]$. Observe that the sequence $\langle N_i[G_\alpha] : i < \gamma \rangle$ is still $\in$-linearly ordered: for $i < j < \gamma$, since $\alpha \in N_j$, N_j can define $\mathbb{A} \restriction \alpha$ and $\dot{G}_\alpha$; since $N_i \in N_j$, $N_i[G_\alpha] \in N_j[G_\alpha]$ by elementarity.

Let $\sigma \in N_0[G_\alpha]$ be a strategy for II witnessing the strategic closure of $\mathbb{Q}_\alpha$ for games of length $2\gamma + 1$. We define a sequence $\langle q_i : i \leq 2\gamma \rangle$ of conditions with $q_{2i} \in N_i[G_\alpha]$ and $q_{2i+1} \in \Gamma_\alpha(N_i[G_\alpha]) \cap N_{i+1}[G_\alpha]$ as follows: we start with a condition $q_1 \in \Gamma_\alpha(N_0[G_\alpha])$ below q_0; note that q_1 can be chosen in $N_1[G_\alpha]$ by elementarity and the definability of the marker set $\Gamma_\alpha(N_0[G_\alpha])$ from α and $N_0[G_\alpha]$. Suppose that i is a successor and $q_{2i-1} \in N_i[G_\alpha]$ is defined. We let $q_{2i} \in N_i[G_\alpha]$ be II's response via σ, and we let q_{2i+1} be a condition in $\Gamma_\alpha(N_i[G_\alpha])$ below q_{2i}, noting that q_{2i+1} can be chosen in $N_{i+1}[G_\alpha]$ by elementarity. In the case that i is a limit, we can choose q_{2i} to be a lower bound for $\langle q_j : j < 2i \rangle$, which exists since II is playing by σ. We observe that if $i < \gamma$, then we can also choose $q_{2i} \in N_i[G_\alpha]$; indeed, this follows since $N_i[G_\alpha]$ is ω-closed in $V[G_\alpha]$ and since $q_j \in N_i[G_\alpha]$ for all $j < 2i$.

Working again in V, this defines our name τ. Setting $p^*(\alpha) = \tau$ finishes the proof. □

The lemma that follows will be used henceforth, often without comment.

Lemma 3.9. *Suppose that $\langle s, p \rangle \in \mathbb{A}^{\mathrm{in}}$, $\alpha \in \mathrm{dom}(p)$, and $\langle t, q \rangle \in \mathbb{A}^{\mathrm{in}} \restriction \alpha$ is a condition which extends $\langle s \restriction \alpha, p \restriction \alpha \rangle$. Then $\langle s, p \rangle$ and $\langle t, q \rangle$ are compatible in $\mathbb{A}^{\mathrm{in}}$. Indeed, $\langle s^*, p^* \rangle$ is a condition in $\mathbb{A}^{\mathrm{in}}$, where $s^* = s \cup t$ and where p^* is the function with $\mathrm{dom}(t) \cup \mathrm{dom}(p)$ such that $p^* \restriction \alpha = t$ and $p^*(\beta) = p(\beta)$ for any $\beta \in \mathrm{dom}(p) \setminus \alpha$.*

Proof. (Sketch) First note that $s^* := s \cup t$ is indeed a condition in $\mathbb{P}(\mathcal{S}, \mathcal{T})$ by part (1) of Lemma 3.5. Conditions (1)-(3) of Definition 3.7 are now immediate. Condition (4) is also clear since all of the conditions under consideration satisfy (4) and since we're amalgamating models below the transitive node $W(\alpha)$. □

The following lemma can be proved using the conclusion of Lemma 3.8, Lemma 3.9, and the strong properness of $\mathbb{P}(\mathcal{S}, \mathcal{T})$ for transitive nodes. We leave the proof to the reader.

Lemma 3.10. *Suppose that the assumptions of Lemma 3.8 hold for all $\alpha \in \mathrm{dom}(\vec{\mathbb{Q}})$.*

1. *If G is generic for $\mathbb{A}$ then $G \restriction \alpha := \{\langle s \restriction \alpha, p \restriction \alpha \rangle : \langle s, p \rangle \in G\}$ is generic for $\mathbb{A} \restriction \alpha$.*
2. *Let H be the upward closure of $\{p(\alpha)[G \restriction \alpha] : \langle s, p \rangle \in G\}$ in $\dot{\mathbb{Q}}_\alpha[G \restriction \alpha]$; then H is generic for $\dot{\mathbb{Q}}_\alpha[G \restriction \alpha]$ over $V[G \restriction \alpha]$*

Lemma 3.11. *Suppose that for all $\alpha \in \mathrm{dom}(\vec{\mathbb{Q}})$, it is forced in $\mathbb{A} \restriction \alpha$ that $\dot{\mathbb{Q}}_\alpha$ is strategically countably closed. Then $\mathbb{A}$ is strategically countably closed. Similarly, with existence of greatest lower bounds for strict descending sequences.*

Proof. For the first claim, a strategy for player II is constructed by playing the (names of the) strategies for the $\dot{\mathbb{Q}}_\alpha$ coordinatewise. Note that condition (4) of Definition 3.7 is maintained since the $\dot{\Gamma}_\alpha$ are forced to be markers, and hence they are forced to be downwards closed. The second claim is similar using the fact that the ordering on $\mathbb{P}(\mathcal{S}, \mathcal{T})$ is reverse inclusion. $\square$

We need one final technical lemma before we can begin the theorem.

Theorem 3.12. *Suppose that for all $\alpha \in \mathrm{dom}(\vec{\mathbb{Q}})$, the assumptions of Lemma 3.8 hold for α and also that $\dot{\mathbb{Q}}_\alpha$ is forced to have greatest lower bounds for strategically decreasing sequences. Suppose $\vec{f} = \langle \dot{f}_\alpha : \alpha \in \mathrm{dom}(\vec{\mathbb{Q}}) \rangle$ is s.t. for all α, $\langle \dot{f}_\alpha(M) : M \in \mathcal{S}(\alpha)^+ \rangle$ is forced in $\mathbb{A} \restriction \alpha$ to be an exact weak residue system for $\dot{\mathbb{Q}}_\alpha$, $\dot{\Gamma}_\alpha$, and $\mathcal{S}(\alpha)^+$. Then for all large enough θ^*, $\mathbb{A}(\mathcal{S}, \mathcal{T}, \vec{\mathbb{Q}}, \vec{\Gamma})$ is proper for all $M^* \prec H(\theta^*)$ with $\vec{\mathbb{Q}}, \vec{\Gamma}, \mathcal{S}, \mathcal{T}, \dot{f} \in M^*$ and $M^* \cap K \in \mathcal{S}$. In particular, if $\mathcal{S}$ is stationary in $\mathcal{P}_{\omega_2}(K)$, then $\mathbb{A}$ preserves ω_2.*

Proof. We will apply Lemma 2.16; we must find, therefore, a marker Δ for $\mathbb{A}$ and the collection $\mathcal{S}$ of models and also a weak residue system for $\mathbb{A}$, $\mathcal{S}$, and Δ. We will first construct Δ. Fix an $M \in \mathcal{S}$, and define

$$\Delta(M) := \{\langle s, p \rangle \in \mathbb{A} : M \in s\} \quad \text{and} \quad \Delta^{\mathrm{in}}(M) := \Delta(M) \cap \mathbb{A}^{\mathrm{in}}$$

Claim 1: $\Delta(M)$ is a marker for $\mathbb{A}$ and $\mathcal{S}$.

Proof. (of Claim 1) It is clear that $\Delta(M)$ is downwards closed, by the definition of $\leq_{\mathbb{P}(\mathcal{S},\mathcal{T})}$. Thus we just need to show that for any $\langle s_0, p_0 \rangle \in M$, there is $\langle s, p \rangle \leq \langle s_0, p_0 \rangle$ with $M \in s$; it suffices to show the result for $\langle s_0, p_0 \rangle \in \mathbb{A}^{\mathrm{in}} \cap M$.[17] By Corollary 2.31 of [7], there is a condition $s \in \mathbb{P}(\mathcal{S}, \mathcal{T})$ containing s_0 and M such that the transitive nodes of s are exactly the transitive nodes as s_0 and such that all new small nodes of s are of the form $W \cap M$ for some transitive $W \in s_0 \subseteq M$. We show how to define p s.t. $\langle s, p \rangle \in \mathbb{A}^{\mathrm{in}} \cap \Delta(M)$ and $\langle s, p \rangle \leq \langle s_0, p_0 \rangle$.

We set $\mathrm{dom}(p) = \mathrm{dom}(p_0)$. Fix $\alpha \in \mathrm{dom}(p)$ so that by (2) of Definition 3.7, $W(\alpha) \in s_0 \subseteq M$. Then since

$$\langle s_0 \restriction \alpha, p_0 \restriction \alpha \rangle \Vdash_\alpha p_0(\alpha) \in \dot{\mathbb{Q}}_\alpha \subseteq W(\alpha)^+[\dot{G}_{\mathbb{A}\restriction\alpha}]$$

and $p_0(\alpha) \in M$, we know that

$$\langle s_0 \restriction \alpha, p_0 \restriction \alpha \rangle \Vdash_\alpha p_0(\alpha) \in (M \cap W(\alpha)^+)[\dot{G}_{\mathbb{A}\restriction\alpha}].$$

So by definition of a marker (and the "maximal principle" for names), we can find an $\mathbb{A} \restriction \alpha$-name τ s.t.

$$\langle s_0 \restriction \alpha, p_0 \restriction \alpha \rangle \Vdash_\alpha \tau \in \dot{\Gamma}_\alpha(M \cap W(\alpha)^+) \wedge \tau \leq_{\dot{\mathbb{Q}}_\alpha} p_0(\alpha).$$

Set $p(\alpha)$ to be τ.

[17]The reason is that every condition $\langle s_0, p_0 \rangle \in \mathbb{A} \cap M$ has an extension in $\mathbb{A}^{\mathrm{in}}$ that belongs to M. This can be seen in cases, depending on whether $\mathrm{dom}(p_0)$ has a largest element, using the definability over M of the posets $\mathbb{A} \restriction \beta$ and $\dot{\mathbb{Q}}_\beta$ from β, and the uniform definability of names for marker sets.

Now it is clear that if $\langle s, p\rangle \in \mathbb{A}^{\text{in}}$, then it is below $\langle s_0, p_0\rangle$. We check, therefore, that it is a condition in $\mathbb{A}^{\text{in}}$. Parts (1), (2), and (3) of Definition 3.7 are clear, so it remains to check (4). Let L and $W = W(\beta)$ be in s satisfying $W \in L$, and we check that

$$\langle s \restriction \beta, p \restriction \beta\rangle \Vdash_\beta p(\beta) \in \dot{\Gamma}_\beta(L \cap W(\beta)^+).$$

We will assume inductively that $\langle s \restriction \beta, p \restriction \beta\rangle$ is a condition in $\mathbb{A}^{\text{in}} \restriction \beta$ below $\langle s_0 \restriction \beta, p_0 \restriction \beta\rangle$. Note that W is a member of s_0 by the remarks in the first paragraph of this proof. So, on the one hand, if $L \in s_0$, then

$$\langle s_0 \restriction \beta, p_0 \restriction \beta\rangle \Vdash_\beta p_0(\beta) \in \dot{\Gamma}_\beta(L \cap W(\beta)^+).$$

However, $\langle s \restriction \beta, p \restriction \beta\rangle$ is below $\langle s_0 \restriction \beta, p_0 \restriction \beta\rangle$ in $\mathbb{A} \restriction \beta$ and forces that $p(\beta)$ is below $p_0(\beta)$; hence it forces that $p(\beta) \in \dot{\Gamma}_\beta(L \cap W(\beta)^+)$ by downwards closure of markers. On the other hand, if $L \notin s_0$, then L must be of the form $W(\gamma) \cap M$ for some $W(\gamma) \in s_0$. Since $\beta \in L \subseteq W(\gamma)$, we conclude that

$$L \cap W(\beta)^+ = (M \cap W(\gamma)) \cap W(\beta)^+ = M \cap W(\beta)^+.$$

However, $p(\beta)$ was a name specifically chosen to be an element of $\dot{\Gamma}_\beta(M \cap W(\beta)^+)$, so by the above equalities, we're done. □

Now we construct a weak residue system, $\langle g_M : M \in \mathcal{S}\rangle$. Fix $M \in \mathcal{S}$; the domain of g_M will be a subset of $\Delta^{\text{in}}(M)$. Fix a condition $\langle s, p\rangle \in \Delta^{\text{in}}(M)$. If for every $\alpha \in \text{dom}(p) \cap M$, there is an $\mathbb{A} \restriction \alpha$-name τ s.t. $\tau \in M$ and

$$\langle s \restriction \alpha, p \restriction \alpha\rangle \Vdash_\alpha \tau = \dot{f}_\alpha(M \cap W(\alpha)^+)(p(\alpha)),$$

then $g_M(s, p)$ will be defined as the pair

$$g_M(s, p) = \langle \text{res}_M(s), \bar{p}\rangle,$$

where $\bar{p}$ has domain equal to $\text{dom}(p) \cap M$, and for each $\alpha \in \text{dom}(\bar{p})$, $\bar{p}(\alpha)$ is some $\tau \in M$ as above. However, if for even a single $\alpha \in \text{dom}(p) \cap M$, there is no such $\mathbb{A} \restriction \alpha$-name τ, then we leave $g_M(s, p)$ undefined.

We now check that $\langle g_M : M \in \mathcal{S}\rangle$ is a weak residue system. We do this through a series of claims.

Claim 2: for all $M \in \mathcal{S}$, $\text{ran}(g_M) \subseteq M \cap \mathbb{A}$.

Proof. (of Claim 2) Fix $M \in \mathcal{S}$. By definition of the system and the countable closure of M, if $g_M(s, p)$ is defined, then it is a member of M. Now let $\langle s, p\rangle \in \text{dom}(g_M)$, and we will check that $\langle \bar{s}, \bar{p}\rangle := g_M(s, p)$ is in $\mathbb{A}$. It is clear that $\langle \bar{s}, \bar{p}\rangle$ satisfies (1)-(3) of Definition 3.7. Moreover, $\langle s, p\rangle \in \mathbb{A}^{\text{in}}$ and $\langle s, p\rangle \leq \langle \bar{s}, \bar{p}\rangle$, with (c) of Definition 3.7 following from the exactness of the residue systems on each coordinate. Thus $\langle \bar{s}, \bar{p}\rangle \in \mathbb{A}$. □

Claim 3: for any $M \in \mathcal{S}$, $\text{dom}(g_M)$ is dense in $\Delta^{\text{in}}(M)$ (and hence in $\Delta(M)$).

Proof. (of Claim 3) First fix, for each $\alpha \in M \cap \text{dom}(\vec{\mathbb{Q}})$, a name $\dot{\sigma}_\alpha$ in M for II's strategy in the exactness game for $\dot{f}_\alpha(M \cap W(\alpha)^+)$, and also fix for all other $\alpha \in \text{dom}(\vec{\mathbb{Q}})$, a name $\dot{\sigma}_\alpha$ witnessing that $\dot{\mathbb{Q}}_\alpha$ is forced to be strategically countably closed. The proof will proceed by fixing a condition in $\Delta^{\text{in}}(M)$ and constructing an ω-sequence of decreasing conditions below it; a lower bound of this sequence will witness the claim. We begin with a single step.

Subclaim 3.1: For any $\langle s,p\rangle \in \Delta^{\text{in}}(M)$ and $\alpha \in M \cap \text{dom}(p)$, there is a condition $\langle t,q\rangle \in \mathbb{A}^{\text{in}} \restriction \alpha$ and there are $\mathbb{A} \restriction \alpha$-names τ' and τ such that

- $\tau \in M$;
- $\langle t,q\rangle \leq_{\mathbb{A}\restriction\alpha} \langle s \restriction \alpha, p \restriction \alpha\rangle$; and
- $\langle t,q\rangle$ forces that τ' is II's response to $p(\alpha)$ via $\dot{\sigma}_\alpha$, and that $\dot{f}_\alpha(M \cap W(\alpha)^+)(\tau') = \tau$.[18]

To see that the subclaim holds, fix a condition $\langle s,p\rangle \in \Delta^{\text{in}}(M)$ and an $\alpha \in M \cap \text{dom}(p)$. We may fix an $\mathbb{A} \restriction \alpha$-name τ' which is forced by $\langle s \restriction \alpha, p \restriction \alpha\rangle$ to be II's reply to $p(\alpha)$ via $\dot{\sigma}_\alpha$. Since $\langle s \restriction \alpha, p \restriction \alpha\rangle$ thereby forces that $\tau' \in \text{dom}(\dot{f}_\alpha(M \cap W(\alpha)^+))$,[19] we know that

$$\langle s \restriction \alpha, p \restriction \alpha\rangle \Vdash \dot{f}_\alpha(M \cap W(\alpha)^+)(\tau') \in M[\dot{G}_{\mathbb{A}\restriction\alpha}].$$

Now we work temporarily in some $\mathbb{A} \restriction \alpha$-generic extension $V[G_{\mathbb{A}\restriction\alpha}]$ s.t. $\langle s \restriction \alpha, p \restriction \alpha\rangle \in G_{\mathbb{A}\restriction\alpha}$. Since $f_\alpha(M \cap W(\alpha)^+)(\tau') \in M[G_{\mathbb{A}\restriction\alpha}]$, we can fix a name $\tau \in M$ s.t.

$$\tau[G_{\mathbb{A}\restriction\alpha}] = f_\alpha(M \cap W(\alpha)^+)(\tau').$$

Let $\langle t,q\rangle$ be any condition in $\mathbb{A}^{\text{in}} \restriction \alpha$ and in $G_{\mathbb{A}\restriction\alpha}$ below $\langle s \restriction \alpha, p \restriction \alpha\rangle$ forcing this. $\langle t,q\rangle$ and τ provide the desired witnesses, finishing the proof of Subclaim 3.1.

We now continue with the proof of Claim 3.[20] Fix a condition $\langle s,p\rangle \in \Delta^{\text{in}}(M)$. We create a descending sequence $\langle\langle s_n,p_n\rangle : n < \omega\rangle$ of conditions in $\mathbb{A}^{\text{in}}$ below $\langle s,p\rangle$, and we use bookkeeping to ensure that the following two conditions are satisfied:

(i) for any α which appears in $\text{dom}(p_i) \cap M$ for some $i < \omega$ (and hence a tail of $i < \omega$), there are infinitely many $k < \omega$ s.t. $\langle s_{k+1}, p_{k+1}\rangle$ is formed by amalgamating $\langle t,q\rangle$ and $\langle s_k,p_k\rangle$, where $\langle t,q\rangle$ is gotten by applying Subclaim 3.1 to $\langle s_k,p_k\rangle$ and $\alpha \in \text{dom}(p_k)$;

(ii) if $\alpha \notin M$, but $\alpha \in \text{dom}(p_i)$ for some i, then there are infinitely many k s.t. $p_{k+1}(\alpha)$ is forced to be II's response to $p_k(\alpha)$ via $\dot{\sigma}_\alpha$.

We leave the details of the bookkeeping to the reader.

Continuing, let $\langle\langle s_n,p_n\rangle : n < \omega\rangle$ be the sequence constructed in the previous paragraph. We aim to create a lower bound $\langle s^*,p^*\rangle$ for this sequence such that $\langle s^*,p^*\rangle \in \text{dom}(g_M)$. Now we know that

$$s^* := \bigcup_{k<\omega} s_k \in \mathbb{P}(\mathcal{S},\mathcal{T}).$$

We define a working part p^* to pair with s^* as follows: let

$$\text{dom}(p^*) := \bigcup_{k<\omega} \text{dom}(p_k).$$

[18]This formulation neglects the fact that II's reply via $\dot{\sigma}_\alpha$ depends on an entire history of the play so far, not only on the last move by I. An accurate formulation would add the history to the subclaim, then in condition (i) below keep track of the entire history of the play constructed at α, and similarly in condition (ii).

[19]Recall that we're playing in the exactness game for $\dot{f}_\alpha(M \cap W(\alpha)^+)$; see Definition 2.10.

[20]The following observation may help to motivate the proof that is to follow. Suppose that we're in the situation in the conclusion of Subclaim 3.1. Then, even though the desired $\mathbb{A} \restriction \alpha$-name $\tau \in M$ has been found, it might be the case that for some $\beta \in \text{dom}(p)$, in the process of refining to $\langle t,q\rangle$ below α, we no longer have a name σ in M forced to be equal to the value of the β-th residue function at $q(\beta)$. This is why we need to construct an ω-sequence of conditions below the given condition and to use bookkeeping.

Suppose that $\gamma \in \mathrm{dom}(p^*)$ and that $p^* \upharpoonright \gamma$ has been constructed in such a way that $\langle s^* \upharpoonright \gamma, p^* \upharpoonright \gamma\rangle$ is a condition in $\mathbb{A}^{\mathrm{in}} \upharpoonright \gamma$ below $\langle s_n \upharpoonright \gamma, p_n \upharpoonright \gamma\rangle$ for all $n < \omega$. We construct $p^*(\gamma)$.

Fix a $k \in \omega$ s.t. $\gamma \in \mathrm{dom}(p_m)$ for all $m \geq k$. If $\gamma \notin M$, then we can let $p^*(\gamma)$ be an $\mathbb{A} \upharpoonright \gamma$-name forced to be a lower bound for $\langle p_m(\gamma) : m \geq k\rangle$; such a name exists by condition (ii) of the bookkeeping above.

Suppose, then, that $\gamma \in M$. We must construct $p^*(\gamma)$ in such a way that condition (4) of Definition 3.7 and the condition for being in $\mathrm{dom}(g_M)$ are met with respect to γ. Since $\gamma \in M$, there are infinitely many $m \geq k$ s.t. $\langle s_{m+1}, p_{m+1}\rangle$ is constructed by amalgamating $\langle s_m, p_m\rangle$ with a condition witnessing Subclaim 3.1 with respect to γ. Let $\langle m_i : i \in \omega\rangle$ enumerate in increasing order all such m, and fix, for each $i < \omega$, an $\mathbb{A} \upharpoonright \gamma$-name τ_i in M s.t. $\langle s_{m_i+1}, p_{m_i+1}\rangle$ forces that $p_{m_i+1}(\gamma)$ is II's reply to $p_{m_i}(\gamma)$ via $\dot{\sigma}_\gamma$ and that $\dot{f}_\gamma(M \cap W(\gamma)^+)(p_{m_i+1}(\gamma)) = \tau_i$. Observe that by definition of the exactness game for $\dot{f}_\gamma(M \cap W(\gamma)^+)$, we know that $\langle s^* \upharpoonright \gamma, p^* \upharpoonright \gamma\rangle$ forces that $\langle p_n(\gamma) : n < \omega\rangle$ has a greatest lower bound, which we set to be $p^*(\gamma)$, such that $p^*(\gamma) \in \mathrm{dom}(\dot{f}_\gamma(M \cap W(\gamma)^+))$ and such that $\dot{f}_\gamma(M \cap W(\gamma)^+)(p^*(\gamma))$ is a greatest lower bound for $\langle \dot{f}_\gamma(M \cap W(\gamma)^+)(p_{m_i+1}(\gamma)) : i < \omega\rangle$.

Now since M is closed under ω-sequences, we know that $\langle \tau_i : i < \omega\rangle \in M$. Thus we can find an $\mathbb{A} \upharpoonright \gamma$-name $\tau_\omega \in M$ s.t. $\mathbb{A} \upharpoonright \gamma$ forces "if $\langle \tau_i : i < \omega\rangle$ is decreasing in $\dot{\mathbb{Q}}_\gamma$ and has a greatest lower bound, then τ_ω is such a greatest lower bound." Consequently, since

$$\langle s^* \upharpoonright \gamma, p^* \upharpoonright \gamma\rangle \Vdash \langle \dot{f}_\gamma(M \cap W(\gamma)^+)(p_{m_i+1}(\gamma)) : i < \omega\rangle = \langle \tau_i : i < \omega\rangle \text{ is decreasing,}$$

(with "decreasing" following from the definition of an exact residue function; see Definition 2.10) we see that $\langle s^* \upharpoonright \gamma, p^* \upharpoonright \gamma\rangle$ forces that both τ_ω and $\dot{f}_\gamma(M \cap W(\gamma)^+)(p^*(\gamma))$ are greatest lower bounds for $\langle \tau_i : i < \omega\rangle$. Hence,

$$\langle s^* \upharpoonright \gamma, p^* \upharpoonright \gamma\rangle \Vdash \tau_\omega = \dot{f}_\gamma(M \cap W(\gamma)^+)(p^*(\gamma)).$$

Since $\tau_\omega \in M$, this shows that the definition of $\mathrm{dom}(g_M)$ is met with respect to γ. It is easy to see that condition (4) of Definition 3.7 is met with respect to γ by the downwards closure of marker sets and by the fact that $\langle s_n, p_n\rangle \in \mathbb{A}^{\mathrm{in}}$ for all $n \in \omega$. With this, the construction of the pair $\langle s^*, p^*\rangle$ and the proof of the claim are complete. □

Claim 4: $\langle g_M : M \in \mathcal{S}\rangle$ satisfies condition (w3) of Definition 2.15.

Proof. (of Claim 4) Suppose that $M \in \mathcal{S}$, $\langle s, p\rangle \in \mathrm{dom}(g_M)$, $\bar{M} \in M \cap \mathcal{S}$, and $\langle t, q\rangle \in M \cap \mathrm{dom}(g_{\bar{M}})$ are such that

$$\langle \bar{s}, \bar{p}\rangle := g_M(s, p) = g_{\bar{M}}(t, q) =: \langle \bar{t}, \bar{q}\rangle.$$

We will show that $\langle s, p\rangle$ and $\langle t, q\rangle$ are compatible in $\mathbb{A}^{\mathrm{in}}$ (they are elements of $\mathbb{A}^{\mathrm{in}}$ since they are in $\mathrm{dom}(g_M)$ and $\mathrm{dom}(g_{\bar{M}})$ respectively). Set s^* to be the weakest condition in $\mathbb{P}(\mathcal{S}, \mathcal{T})$ witnessing that s and t are compatible. We define p^* as follows: the domain of p^* will be $\mathrm{dom}(p) \cup \mathrm{dom}(q)$. If $\gamma \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$, then let $p^*(\gamma) = p(\gamma)$. If $\gamma \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$, then let $p^*(\gamma)$ be some $\mathbb{A} \upharpoonright \gamma$-name forced to be a condition in $\dot{\mathbb{Q}}_\gamma$ below $p(\gamma)$ and $q(\gamma)$, if such a condition exists (and the maximal condition otherwise). If $\gamma \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$, then we take $p^*(\gamma)$ to be an $\mathbb{A} \upharpoonright \gamma$-name for a condition which is below $q(\gamma)$ in $\dot{\mathbb{Q}}_\gamma$ which is forced to satisfy (4) of Definition 3.7 with respect to γ and the nodes in s^*, if such a condition exists.

We claim that $\langle s^*, p^*\rangle$ is a condition in $\mathbb{A}^{\text{in}}$ below $\langle s, p\rangle$ and $\langle t, q\rangle$. It is clear that $\langle s^*, p^*\rangle$ satisfies conditions (1)-(3) of Definition 3.7. We prove by induction on $\gamma \in \text{dom}(p^*)$ that the following hold:

(a) for all small $N \in s^*$ with $\gamma \in N$, $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle \Vdash_\gamma p^*(\gamma) \in \dot{\Gamma}_\gamma(N \cap W(\gamma)^+)$;
(b) $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ forces that $p^*(\gamma)$ is below whichever (or both) of $p(\gamma)$ or $q(\gamma)$ is defined.

Observe that by induction, condition (a) holds at $\bar{\gamma} < \gamma$. This implies in particular that $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ is a condition in $\mathbb{A} \restriction \gamma$, a fact that is used implicitly in both conditions (a) and (b). Similarly, by induction condition (b) holds at $\bar{\gamma} < \gamma$. This implies that $\langle s^*, \restriction \gamma, p^* \restriction \gamma\rangle$ extends both $\langle s \restriction \gamma, p \restriction \gamma\rangle$ and $\langle t \restriction \gamma, q \restriction \gamma\rangle$.

To prove (a) and (b), fix an $N \in s^*$ with $\gamma \in N$; we have a number of cases. First suppose that $\gamma \in \text{dom}(p)\setminus \text{dom}(q)$. Observe that $N \notin M$: otherwise, since $\gamma \in N \subseteq M$, we'd have

$$\gamma \in M \cap \text{dom}(p) = \text{dom}(\bar{p}) = \text{dom}(\bar{q}) \subseteq \text{dom}(q),$$

contradicting our assumption about γ. Thus we know that $N \notin M$. This implies that $N \notin t$, and hence N is either an old node in $s\setminus t$, or a new node added in the process of closing $s \cup t$ under intersections to form s^*. In the first case, (a) follows since $p(\gamma) = p^*(\gamma)$ and $\langle s, p\rangle \in \mathbb{A}^{\text{in}}$. In the second case, N is a new node, and therefore by Corollary 2.31(3) of [7], $N = L \cap W(\beta)$ for some transitive $W(\beta) \in t$ and small $L \in s$. Since $\gamma \in N \subseteq W(\beta)$, $W(\gamma)^+ \subseteq W(\beta)$. Thus

$$N \cap W(\gamma)^+ = (L \cap W(\beta)) \cap W(\gamma)^+ = L \cap W(\gamma)^+.$$

Consequently, as $\langle s, p\rangle \in \mathbb{A}^{\text{in}}$, (a) holds. Observe that in either case, (b) is immediate since $q(\gamma)$ is not defined and $p^*(\gamma) = p(\gamma)$.

Second, suppose that $\gamma \in \text{dom}(p) \cap \text{dom}(q)$. Since the residue system $\langle \dot{f}_\gamma(L) : L \in \mathcal{S}(\gamma)^+\rangle$ is forced to satisfy (w3) of Definition 2.15, and since $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ extends both $\langle s \restriction \gamma, p \restriction \gamma\rangle$ and $\langle t \restriction \gamma, q \restriction \gamma\rangle$, we know that $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle \Vdash_\gamma p(\gamma)$ is compatible in $\dot{\mathbb{Q}}_\gamma$ with $q(\gamma)$. Thus $p^*(\gamma)$ is a name forced to be a lower bound, and (b) is secured. For condition (a), the result follows by the fact that $\langle s, p\rangle, \langle t, q\rangle$ are conditions in $\mathbb{A}^{\text{in}}$ and by the characterization of the new nodes in s^* (as in the previous paragraph).

The final case is when $\gamma \in \text{dom}(q)\setminus \text{dom}(p)$. We claim that $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ forces that there is a condition in $\dot{\mathbb{Q}}_\gamma$ below $q(\gamma)$ which satisfies (a); showing this will suffice since (b) holds as a result ($p(\gamma)$ is not defined). If $N \in M$, then $N \in s^* \cap M = t$, with equality holding by Corollary 2.31(2) of [7]. Since $\langle t, q\rangle$ is a condition in $\mathbb{A}^{\text{in}}$, $\langle t \restriction \gamma, q \restriction \gamma\rangle \Vdash q(\gamma) \in \dot{\Gamma}_\gamma(N \cap W(\gamma)^+)$, and using the downward closure of markers, this automatically implies $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle \Vdash p^*(\gamma) \in \dot{\Gamma}_\gamma(N \cap W(\gamma)^+)$ for any $p^*(\gamma)$ extending $q(\gamma)$, as desired. We show how to secure (a) for the nodes $N \in s^*\setminus M$. By Lemma 3.6, we know that the set of nodes

$$Z := \left\{L \cap W(\gamma)^+ : \gamma \in L,\ L \in s^* \text{ is small}\right\}$$

is $\in$-linearly ordered; observe that $M \cap W(\gamma)^+$ belongs to Z. Arguing as in Lemma 3.8, but starting from $q(\gamma)$ and working only through the nodes of Z from $M \cap W(\gamma)^+$ upwards, we can create an $\mathbb{A} \restriction \gamma$-name $p^*(\gamma)$ forced by $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ to be in each of their marker sets and to extend $q(\gamma)$. $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle$ also forces $p^*(\gamma)$ into the marker sets for the models in Z before $M \cap W(\gamma)^+$, since each of these nodes belongs to t and $p^*(\gamma)$ extends $q(\gamma)$. Thus $\langle s^* \restriction \gamma, p^* \restriction \gamma\rangle \Vdash p^*(\gamma) \in \dot{\Gamma}_\gamma(L \cap W(\gamma)^+)$ for all small $L \in s^*$ with $\gamma \in L$, as desired for condition (a). □

Since we've now produced a marker Δ for $\mathbb{A}$ and $\mathcal{S}$ as well as a weak residue system for Δ and $\mathbb{A}$, we're done. □

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