Σ_1-partition properties

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Introduction

If X is a set, then we let $[X]^2$ denote the set consisting of all two-element subsets of X.

Given a function c with domain $[X]^2$, we say that a subset H of X is *c-homogeneous* if $c \restriction [H]^2$ is constant.

Classical results of Erdős and Tarski show that an uncountable cardinal κ is weakly compact if and only if for every colouring $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogenoues subset of κ of cardinality κ.

Colourings witnessing failures of weak compactness are usually constructed using κ-Aronszajn trees, non-reflecting subsets of κ or wellorderings of power sets of cardinals smaller than κ.

The work presented in this paper is motivated by the question whether such colourings can be simply definable, i.e. whether they can be defined by formulas of low quantifier complexity that use parameters of low hereditary cardinality.

Definition

Given $n < \omega$ and sets $z_0, \ldots, z_{n-1}$, a class X is $\Sigma_n(z_0, \ldots, z_{n-1})$-*definable* if there is a Σ_n-formula $\varphi(v_0, \ldots, v_n)$ with

$$X \;=\; \{x \mid \varphi(x, z_0, \ldots, z_{n-1})\}.$$

Definition

An infinite cardinal κ *has the* $\Sigma_n(z)$*-partition property* if, for every $\Sigma_n(\kappa, z)$-definable function $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogeneous set of cardinality κ.

Definition

An infinite regular cardinal κ is $\boldsymbol{\Sigma}_n$-weakly compact if κ has the $\Sigma_n(z)$-partition property for every $z \in \mathrm{H}(\kappa)$.

Summary of results

The work presented in this talk focusses on the question whether the validity or failure of $\mathbf{\Sigma}_n$-weak compactness of certain uncountable regular cardinals is decided by (canonical extensions of) $\mathbf{ZFC}$.

Our first result shows that this is the case for ω_1.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- *Bounded Martin's Maximum holds and the nonstationary ideal on ω_1 is precipitous.*
- *Woodin's Axiom $(*)$ holds.*

Then ω_1 is $\mathbf{\Sigma}_1$-weakly compact.

We list a number of complementary results:

- The existence of a Woodin cardinal alone does not imply that ω_1 is $\mathbf{\Sigma}_1$-weakly compact.
- If the *Bounded Proper Forcing Axiom* holds, then ω_2 is not $\mathbf{\Sigma}_1$-weakly compact.
- If κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a partial order $\mathbb{P}$ with the following properties:
  - $\mathbb{P}$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most κ^+.
  - If G is $\mathbb{P}$-generic over V, then κ^+ is not $\mathbf{\Sigma}_1$-weakly compact in $\mathrm{V}[G]$.

- If κ is a $\mathbf{\Sigma}_n$-weakly compact cardinal, then κ is an inaccessible $\mathbf{\Sigma}_n$-weakly compact cardinal in L.
- If ν is an infinite regular cardinal, $\kappa > \nu$ is weakly compact and G is $\mathrm{Col}(\nu, {<}\kappa)$-generic over V, then κ is $\mathbf{\Sigma}_n$-weakly compact for all $0 < n < \omega$.

In particular, the $\mathbf{\Sigma}_1$-weak compactness of ω_2 is independent of large cardinal axioms.

Next, we discuss examples of inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinals that are not weakly compact.

Theorem

Let κ be a weakly compact cardinal. Then every $\mathbf{\Pi}^1_1$-statement that holds in V_κ reflects to an inaccessible Σ_1-weakly compact cardinal less than κ.

Theorem

If κ is a regular cardinal that is a stationary limit of ω_1-iterable cardinals, then κ is $\mathbf{\Sigma}_1$-weakly compact.

Since measurable cardinals are ω_1-iterable and Woodin cardinals are stationary limits of measurable cardinals, this result shows that the first Woodin cardinal is an example of an inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinal that is not weakly compact.

Now, we want to measure the consistency strength of $\mathbf{\Sigma}_1$-weak compactness by determining the position of the least $\mathbf{\Sigma}_1$-weakly compact cardinal in the large cardinal hierarchy of the constructible universe L. The above result already shows that this cardinal is strictly smaller than the first weakly compact cardinal. The following result yields a lower bound.

Theorem

If $\mathrm{V} = \mathrm{L}$ holds, then every $\mathbf{\Sigma}_1$-weakly compact cardinal is a hyper-Mahlo cardinal.

The proof of this result relies on Todorčević's method of *walks on ordinals* and failures of simultaneous reflection of definable stationary subsets.

Finally, we consider $\mathbf{\Sigma}_n$-weak compactness for $n > 1$.

Theorem

Assume that $\Psi(v_0, v_1)$ *is a formula that defines a global wellordering* $\lhd$ *of* V *of order-type* On *such that the class*

$$I \;=\; \{\{x \mid \Psi(x,y)\} \mid y \in \mathrm{V}\}$$

of all initial segments of $\lhd$ *is* Σ_n*-definable for some* $1 < n < \omega$*. Then all* $\mathbf{\Sigma}_n$*-weakly compact cardinals are weakly compact.*

Note that the existence of such a *good global* Σ_2*-wellordering* is relative consistent with the existence of very large large cardinals (like *supercompact cardinals*) and strong forcing axioms (like *Martin's Maximum*). In combination with the above results, this shows that such extensions of $\mathbf{ZFC}$ do not decide the $\mathbf{\Sigma}_2$-weak compactness of uncountable regular cardinals.

Some basic results

By carefully reviewing the proof of the classical *Ramification Lemma*, it is possible to derive the following *definability version* of that result:

Lemma

Given a set z and $0 < n < \omega$, the following statements are equivalent for every infinite regular cardinal κ:

- *κ has the $\Sigma_n(z)$-partition property.*
- *If $\iota : \kappa \longrightarrow {}^{<\kappa}2$ is a $\Sigma_n(\kappa, z)$-definable injection with the property that $\mathrm{ran}(\iota)$ is a subtree of ${}^{<\kappa}2$ of height κ, then there is a cofinal branch through this subtree.*

This lemma allows us to show that $\mathbf{\Sigma}_1$-weakly compact cardinal are inaccessible in L.

Definition

Given $n < \omega$ and sets $z_0, \ldots, z_{n-1}$, a wellordering $\lhd$ of a set X is a *good* $\Sigma_n(z_0, \ldots, z_{n-1})$*-wellordering* if the set

$$I(\lhd) \; = \; \{\{y \mid y \lhd x\} \mid x \in X\}$$

of all proper initial segments of $\lhd$ is $\Sigma_n(z_0, \ldots, z_{n-1})$-definable.

Lemma

Let $\nu < \kappa \leq 2^\nu$ be infinite cardinals with the property that there is a good $\Sigma_n(\kappa, z)$-wellordering $\lhd$ of $\mathcal{P}(\nu)$. Then κ does not have the $\Sigma_n(z)$-partition property.

Corollary

If $\mathrm{V} = \mathrm{L}$ holds, then all $\mathbf{\Sigma}_1$-weakly compact cardinals are inaccessible.

Next, we show that the first $\mathbf{\Sigma}_1$-weakly compact cardinal is much smaller than the first weakly compact cardinal.

Note that the first $\mathbf{\Sigma}_2$-weakly compact cardinal can be the first weakly compact cardinal.

Theorem

Let κ be a weakly compact cardinal. Then every $\mathbf{\Pi}^1_1$-statement that holds in V_κ reflects to an inaccessible Σ_1-weakly compact cardinal less than κ.

Proof.

Fix a Π^1_1-formula $\Psi(v)$ and $A \subseteq \kappa$ with $\mathrm{V}_\kappa \models \Psi(A)$. Pick an elementary submodel M of $\mathrm{H}(\kappa^+)$ of cardinality κ with $\mathrm{H}(\kappa) \cup \{A\} \subseteq M$ and ${}^{<\kappa}M \subseteq M$. By the *Hauser characterization* of weak compactness, there is a transitive set N and an elementary embedding $j : M \longrightarrow N$ with critical point κ and $M \in N$. Then κ is inaccessible in N, $A = j(A) \cap \kappa$, $\mathrm{V}_\kappa \in N$ and Π^1_1-downwards absoluteness implies that $\mathrm{V}_\kappa \models \Psi(A)$ holds in N.

The above construction ensures that κ is weakly compact in M and all Σ_1-formulas with parameters in M are absolute between M and N. In particular, every function $c : [\kappa]^2 \longrightarrow 2$ that is definable in N by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$ is definable in M by the the same formula and hence there is a c-homogeneous set of cardinality κ in $M \subseteq N$.

This shows that κ is $\mathbf{\Sigma}_1$-weakly compact in N. With the help of a universal Σ_1-formula this yields the statement of the theorem. $\square$

The Σ_1-club property

We show how the above results on the $\boldsymbol{\Sigma}_1$-weak compactness of ω_1 and certain large cardinals can be derived.

Definition

Given $0 < n < \omega$, an uncountable regular cardinal κ has the $\boldsymbol{\Sigma}_n$-*club property* if every subset x of κ with the property that the set $\{x\}$ is $\Sigma_n(\kappa, z)$ for some $z \in \mathrm{H}(\kappa)$ either contains a club subset of κ or is disjoint from such a set.

Lemma

Given $0 < n < \omega$, *if an uncountable regular cardinal* κ *has the* $\boldsymbol{\Sigma}_n$*-club property, then* κ *is* $\boldsymbol{\Sigma}_n$*-weakly compact.*

Sketch of the proof.

Fix $z \in \mathrm{H}(\kappa)$ and a $\Sigma_n(\kappa, z)$-definable injection $\iota : \kappa \longrightarrow {}^{<\kappa}2$ with the property that $\mathbb{T} = \mathrm{ran}(\iota)$ is a subtree of ${}^{<\kappa}2$ of height κ.

Given $\beta < \kappa$, define

$$D_\beta \ = \ \{\gamma < \kappa \mid \iota(\beta) \subsetneq \iota(\gamma)\}.$$

Note that our assumptions imply that the set $\{D_\beta\}$ is $\Sigma_n(\kappa, \beta, z)$-definable for all $\beta < \kappa$. In particular, the $\mathbf{\Sigma}_n$-club property implies that sets of the form D_β either contain a club subset of κ or they are disjoint from such a subset.

By induction, we construct a sequence $\langle \beta_\alpha \mid \alpha < \kappa \rangle$ such that the following statements hold for all $\alpha < \kappa$:

- $\mathrm{dom}(\iota(\beta_\alpha)) = \alpha$ and $\iota(\beta_{\bar{\alpha}}) \subseteq \iota(\beta_\alpha)$ for all $\bar{\alpha} < \alpha$.
- The set D_{β_α} contains a club subset of κ.

Then $x = \bigcup\{\iota(\beta_\alpha) \mid \alpha < \kappa\}$ is a cofinal branch through $\mathbb{T}$. □

The above theorem about the $\mathbf{\Sigma}_1$-weak compactness of ω_1 is now a direct consequence of the following result.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- *Bounded Martin's Maximum holds and the nonstationary ideal on ω_1 is precipitous.*
- *Woodin's Axiom $(*)$ holds.*

Then ω_1 has the $\mathbf{\Sigma}_1$-club property.

We will present a simplified version of the proof of the first implication that uses results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$.

Proposition

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then ω_1 has the $\mathbf{\Sigma}_1$-club property.

Proof.

Given a Σ_1-formula $\varphi(v_0, v_1, v_2)$, a bistationary subset A of ω_1 and $z \in \mathbb{R}$, assume that A is the unique subset of ω_1 with $\varphi(A, \omega_1, z)$.

Let G be $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$. By the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$, there is $B \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})[G]}$ such that B is bistationary subset of ω_1 in $\mathrm{L}(\mathbb{R})[G]$ and B is the unique subset of ω_1 with $\varphi(B, \omega_1, z)$ in $\mathrm{L}(\mathbb{R})[G]$. Since the partial order $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, we have $B \in \mathrm{L}(\mathbb{R})$.

Since our assumptions imply that AD holds in $\mathrm{L}(\mathbb{R})$ and therefore the clubfilter on ω_1 is an ultrafilter, there is a club subset C of ω_1 such that either $C \subseteq B$ or $B \cap C = \emptyset$. But this contradicts the bistationarity of B in $\mathrm{L}(\mathbb{R})[G]$. $\square$

The following lemma allows us to derive the above implication from the weaker large cardinal assumption.

Lemma (L.–Schindler–Schlicht)

Assume that $M_1^{\#}(A)$ exists for every $A \subseteq \omega_1$. Pick a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in \mathbb{R}$.

- *If there is a stationary subset x of ω_1 such that $\varphi(\omega_1, x, z)$ holds, then there is an element y of the club filter on ω_1 such that $\varphi(\omega_1, y, z)$ holds.*
- *If there is a costationary subset x of ω_1 such that $\varphi(\omega_1, x, z)$ holds, then there is an element y of the non-stationary ideal on ω_1 such that $\varphi(\omega_1, y, z)$ holds.*

The proof of this result uses iterated generic ultrapowers and Woodin's countable stationary tower forcing.

Next, we consider examples of inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinals that are not weakly compact.

Definition (Sharpe & Welch)

Let κ be an uncountable cardinal.

- A *weak κ-model* is a transitive model M of ZFC$^-$ of size κ with $\kappa \in M$.
- The cardinal κ is *ω_1-iterable* if for every subset A of κ there is a weak κ-model M and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and $\langle M, \in, U \rangle$ is ω_1-iterable.

Theorem

If κ is a regular cardinal that is a stationary limit of ω_1-iterable cardinals, then κ has the $\mathbf{\Sigma}_1$-club property.

Proof of the Theorem.

Assume that there is a Σ_1-formula $\varphi(v_0, v_1, v_2)$, a subset A of κ and $z \in \mathrm{H}(\kappa)$ such that A is the unique subset of κ with $\varphi(A, \kappa, z)$.

Take a continuous chain $\langle M_\alpha \mid \alpha < \kappa\rangle$ of elementary submodels of $\mathrm{H}(\kappa^+)$ of cardinality less than κ with $\mathrm{tc}(\{z\}) \cup \{\kappa, A\} \in M_0$, $\varphi(A, \kappa, z)^{M_0}$ and $M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$.

Then there is $\nu < \kappa$ ω_1-iterable with $\nu = M_\nu \cap \kappa = |M_\nu|$. Let B be a subset of ν that codes the transitive collapse of M_ν. Pick a weak ν-model N_0 and a weakly amenable N_0-ultrafilter U on ν such that $B \in N_0$ and $\langle N_0, \in, U\rangle$ is ω_1-iterable. Then $\varphi(A \cap \nu, \nu, z)$ holds in N_0. Let

$$\langle\langle N_\alpha \mid \alpha \leq \kappa\rangle,\ \langle j_{\bar{\alpha},\alpha} : N_{\bar{\alpha}} \longrightarrow N_\alpha \mid \bar{\alpha} \leq \alpha \leq \kappa\rangle\rangle$$

be an iteration of $\langle N_0, \in, U\rangle$. Then $\varphi(j_{0,\kappa}(A \cap \nu), \kappa, z)$ holds and hence $A = j_{0,\kappa}(A \cap \nu)$. Set $C = \{j_{0,\alpha}(\nu) \mid \alpha < \kappa\}$ club in κ.

In this situation, we know that $A \cap \nu \in U$ implies that $C \subseteq A$ and $A \cap \nu \notin U$ implies that $A \cap C = \emptyset$. $\square$

Remark

If $\mathrm{V} = \mathrm{L}$ and κ is an uncountable regular cardinal, then there is a bistationary subset x of κ such that $\{x\}$ is $\Sigma_1(\kappa)$-definable. Such subsets can be constructed from the canonical $\Diamond_\kappa$-sequence in L, using the facts that this sequence is definable over $\langle \mathrm{L}_\kappa, \in \rangle$ by a formula without parameters and the set $\{\mathrm{L}_\kappa\}$ is $\Sigma_1(\kappa)$-definable.

Σ_1-weakly compact cardinals in the constructible universe

Remember that, given a cardinal κ and an ordinal α, we say that κ is an α-*Mahlo* if κ is a Mahlo cardinal and for every $\bar{\alpha} < \alpha$, the set $\{\nu < \kappa \mid \nu \textit{ is an } \bar{\alpha}\textit{-Mahlo cardinal}\}$ is stationary in κ.

Finally, we say that κ is *hyper-Mahlo* if κ is a κ-Mahlo cardinal.

Theorem

If $\mathrm{V} = \mathrm{L}$ *holds, then every* $\boldsymbol{\Sigma}_1$*-weakly compact cardinal is a hyper-Mahlo cardinal.*

The proof of this result relies on the following lemma.

Lemma

Assume that $\mathrm{V} = \mathrm{L}$. *Let* κ *be an inaccessible cardinal and let* $\langle S_\alpha \mid \alpha < \lambda \rangle$ *be a sequence of stationary subsets of* κ *with* $\lambda < \kappa$ *such that the following statements hold:*

- *The set* $\{\langle \alpha, \gamma \rangle \mid \alpha < \lambda, \ \gamma \in S_\alpha\}$ *is* $\Delta_1(\kappa, z)$*-definable.*
- *The set*

  $$\{\nu \in \mathrm{Lim} \cap \kappa \mid \mathrm{cof}(\nu) = \nu, \ S_\alpha \cap \nu \text{ is stationary in } \nu \text{ for all } \alpha < \lambda\}$$

  is not stationary in κ.

Then κ *does not have the* $\Sigma_1(z)$*-partition property.*

Sketch of the proof.

Let C denote the $<_{\mathrm{L}}$-least club in κ with the property that for every regular $\nu \in C$, there is an $\alpha < \lambda$ with the property that $S_\alpha \cap \mathrm{Lim}(C) \cap \nu$ is not stationary in ν.

Let $\vec{C} = \langle C_\gamma \mid \gamma < \kappa \rangle$ be the unique C-sequence of length κ with the property that for every $\gamma \in \mathrm{Lim} \cap \kappa$, the club C_γ is $<_{\mathrm{L}}$-minimal with the following properties:

- *If γ is singular, then $\mathrm{cof}(\gamma) < \min(C_\gamma)$.*
- *If $\gamma = \mu^+$ for a cardinal μ, then $C_\gamma = (\mu, \gamma)$.*
- *If γ is an inaccessible cardinal, then there is $\alpha(\gamma) < \lambda$ with $C_\gamma \cap S_{\alpha(\gamma)} \cap \mathrm{Lim}(C) = \emptyset$.*

Then the set $\{\vec{C}\}$ is $\Sigma_1(\kappa, z)$-definable.

Sketch of the proof (cont.)

Using techniques developed by Todorčević, we can construct a slim tree $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$ of height κ with the following properties:

- $\mathbb{T}$ is a $\Delta_1(\kappa, z)$-definable subset of $\mathrm{H}(\kappa)$.
- $\mathbb{T}$ has a cofinal branch if and only if there is $\xi < \kappa$ and a club D in κ such that for every $\xi < \gamma \in \mathrm{Lim}(D)$, there is a $\gamma \leq \delta(\gamma) < \kappa$ with

$$D \cap \gamma \ = \ C_{\delta(\gamma)} \cap [\xi, \gamma).$$

Assume that D witnesses that $\mathbb{T}$ has a cofinal branch. Then there is a club $D_* \subseteq \mathrm{Lim}(D)$ consisting of strong limit cardinals such that $\delta(\gamma)$ is inaccessible for every $\gamma \in C$. But this yields an $\alpha < \lambda$ with $\alpha = \alpha(\delta(\gamma))$ for stationary-many $\gamma \in D_*$ and hence $D \cap S_\alpha \cap \mathrm{Lim}(C) = \emptyset$, a contradiction.

This shows that $\mathbb{T}$ is a κ-Aronszajn tree and we can use this tree to construct a counterexample to the $\Sigma_1(z)$-partition property. $\square$

Thank you for listening!