

The current state of the Souslin problem

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Recent bibliography

1. A. M. Brodsky and A. R., [A Microscopic approach to Souslin-tree constructions. Part I](#), submitted December 2015.
2. A. M. Brodsky and A. R., [More notions of forcing add a Souslin tree](#), submitted July 2016.
3. A. M. Brodsky and A. R., [Reduced powers of Souslin trees](#), *Forum Math. Sigma*, 5(e2): 1-82, 2017.
4. A. R., [Higher Souslin trees and the GCH, revisited](#), *Adv. Math.*, 311(c): 510-531, 2017.
5. A. M. Brodsky and A. R., [Distributive Aronszajn trees](#), to be submitted.

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The Souslin problem

Theorem (Cantor, end of 1800)

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Corollary

Any **separable** dense complete linear order with no endpoints is isomorphic to (\mathbb{R}, \leq) .

Recall

A linear order is separable if it admits a countable dense set.

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Definition

A linear order is said to satisfy the countable chain condition (**ccc**) if any collection of pairwise disjoint open intervals is countable.

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Any separable linear order is ccc:

Let \mathcal{I} be a collection of pairwise disjoint nonempty open intervals.
Let D be countable and dense.

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Any separable linear order is ccc:

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The Souslin problem

Is there a ccc linear order which is not separable?

Souslin trees

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A **tree** is a poset $(T, <)$ satisfying that $x_{\downarrow} := \{y \in T \mid y < x\}$ is well-ordered by $<$ for any node $x \in T$.

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Note (Sierpiński, 1933)

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The following are equivalent:

- ▶ There is a ccc linear order which is not separable;
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Do Souslin trees exist?

From the Amer. Math. Monthly, 76(10), p. 1113, 1969

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“Souslin’s conjecture sounds simple. Anyone who understands the meaning of countable and uncountable can “work” on it. It is in fact very tricky. There are standard patterns one builds. There are standard errors in judgement one makes. And there are standard not-quite-counter-examples which almost everyone who looks at the problem happens upon.”

Importance

Souslin trees have numerous applications. To mention a few:

Theorem (Kurepa, 1952)

If there exists a Souslin tree, then there exist ccc Boolean algebras $\mathbb{B}_0, \mathbb{B}_1$ whose free product $\mathbb{B}_0 \oplus \mathbb{B}_1$ is not ccc.

Importance

Souslin trees have numerous applications. To mention a few:

Theorem (Rudin, 1955)

If there exists a Souslin tree, then there exists a normal Hausdorff topological space \mathbb{X} for which $\mathbb{X} \times [0, 1]$ is not normal.

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Theorem (Baumgartner-Malitz-Reinhardt, 1970)

If there exists a Souslin tree, then there exists a graph \mathcal{G} of size and chromatic number \aleph_1 such that, in some cardinality-preserving forcing extension, \mathcal{G} is countably chromatic.

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Theorem (Raghavan-Todorcevic, 2014)

If there exists a Souslin tree, then $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ fails.

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Theorem (Raghavan-Todorćević, 2014)

If there exists a Souslin tree, then $\omega_1 \rightarrow (\omega_1, \omega + 2)^2$ fails.

Recall (Erdős-Dushnik-Miller, 1941)

$\omega_1 \rightarrow (\omega_1, \omega + 1)^2$ holds.

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Theorem (Aronszajn, 1934)

*There exists a **fake Souslin tree**.*

That is, a tree of size \aleph_1 having no chains or levels of size \aleph_1 .

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A Souslin tree

A tree of size \aleph_1 having no chains or **antichains** of size \aleph_1 .

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Lemma (König, 1927)

There are no \aleph_0 -Aronszajn trees.

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Theorem (Specker, 1949)

CH entails the existence of an \aleph_2 -Aronszajn tree.

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Theorem (Aronszajn, 1934)

There exists an \aleph_1 -Aronszajn tree.

Theorem (Specker, 1949)

CH entails the existence of an \aleph_2 -Aronszajn tree.

Furthermore, if $\lambda^{<\lambda} = \lambda$, then there exists a λ^+ -Aronszajn tree.

Souslin's problem and Set Theory

The Souslin problem was resolved at the end of the 1960's:

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At the 1970's, Jensen developed a framework for **iterating forcing without adding reals** (aka, NNR) and used it to prove that the nonexistence of a Souslin tree is consistent with $ZFC+GCH$.

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At the 1970's, Jensen developed a framework for **iterating forcing without adding reals** (aka, NNR) and used it to prove that the nonexistence of a Souslin tree is consistent with $ZFC+GCH$.

At the 1980's, Shelah proved that in Cohen's original model of $ZFC+\neg CH$, there exists a Souslin tree.

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Remarkably, the Souslin Problem is responsible for the discovery of: Iterated forcing, Jensen's \diamond and \square combinatorial principles, forcing axioms (e.g., Martin's Axiom), and NNR iterations (that played a role in refuting some false claims about the Whitehead problem).

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The higher trees are as useful as their little siblings. For instance:

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If there is a κ -Souslin tree, then there is a rainbow-triangle-free strong counterexample to Ramsey's theorem at the level of κ .

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Specifically, there is a symmetric coloring $c : [\kappa]^2 \rightarrow \kappa$ such that:

- ▶ *every $A \subseteq \kappa$ of size κ is omnichromatic, i.e., $c \upharpoonright [A]^2$ is onto κ ;*

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- ▶ for every $\alpha < \beta < \gamma < \kappa$, $|\{c(\alpha, \beta), c(\alpha, \gamma), c(\beta, \gamma)\}| < 3$.

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Theorem (Jensen, 1972)

If $V = L$, then for every regular uncountable cardinal κ , TFAE:

- ▶ There exists a κ -Aronszajn tree;
- ▶ There exists a κ -Souslin tree;
- ▶ κ is not weakly compact.

Recall

κ is weakly compact if it satisfies the generalized Ramsey partition relation: $\kappa \rightarrow (\kappa)_2^2$.

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Theorem (Mitchell-Silver, 1973)

The nonexistence of an \aleph_2 -Aronszajn tree is equiconsistent with the existence of a weakly compact cardinal.

Higher Souslin problem

Recall (Specker, 1949)

CH implies the existence of an \aleph_2 -Aronszajn tree.

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Recall (Jensen, 1974)

CH is consistent with the nonexistence of an \aleph_1 -Souslin tree.

The \aleph_2 -Souslin problem

The above-mentioned results crystallized the following question:

Question (folklore, 1970's)

Does GCH entail the existence of an \aleph_2 -Souslin tree?

If not, is the consistency strength of a negative answer a weakly compact cardinal?

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If GCH holds and there are no \aleph_2 -Souslin trees, then there exists a Mahlo cardinal in L .

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Note (Hanf, 1964)

If κ is weakly compact, then there are (stationarily many) Mahlo cardinals below κ .

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Theorem (Gregory, 1976)

If GCH holds and there are no \aleph_2 -Souslin trees, then $L \models (\aleph_2)^V$ is a Mahlo cardinal.

The \aleph_2 -Souslin problem (cont.)

From the Kanamori-Magidor 1978 survey article (p. 261):

The consistency problem for SH_κ when $\kappa > \omega_1$ seems to be much more difficult, especially if we want to retain the GCH. To bring matters into focus, we make some remarks which recall and amplify §21. First of all, Jensen[1972] had actually established that in L , weak compactness for κ is equivalent to SH_κ , for regular κ . We are interested in SH_κ for small κ , and the Mitchell-Silver model cited in §21 certainly satisfied SH_{ω_2} , as there were not even any ω_2 -Aronszajn trees in that model. However, $2^\omega = \omega_2$ held in that model, and in fact a classical result of Specker[1951] as cited in §5 necessitates something like this: if $2^\omega = \omega_1$, then there is an ω_2 -Aronszajn tree. No such result seems available for ω_2 -Souslin trees, so the focal problem in this area is to get SH_{ω_2} and the GCH to hold.

This problem has been extensively investigated by Gregory[1976] who established in particular that: If $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $E_{\omega_2}^\omega$ hold, then SH_{ω_2} is false, i.e. there is an ω_2 -Souslin tree. Hence, if we want SH_{ω_2} and the GCH to hold, we need to guarantee the failure of $E_{\omega_2}^\omega$. As pointed out in §21, this necessitates at least the consistency strength of the existence of a Mahlo cardinal, and very likely, of a weakly compact cardinal.

The \aleph_2 -Souslin problem (cont.)

Theorem (Gregory, 1976)

If GCH holds and $L \models (\aleph_2)^V$ is not a Mahlo cardinal, then there exists an \aleph_2 -Souslin tree.

Theorem (2016)

If GCH holds and $L \models (\aleph_2)^V$ is not weakly compact, then there exists an \aleph_2 -Souslin tree.

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If GCH holds and $L \models (\aleph_2)^V$ is not weakly compact, then there exists an \aleph_2 -Souslin tree.

I recently found an email I sent to Gitik on Nov/2009 with an alleged proof of the above theorem. This was my first false proof of the theorem, and there were certainly a few more along the way.

The \aleph_2 -Souslin problem (cont.)

Theorem (Gregory, 1976)

If GCH holds and $L \models (\aleph_2)^V$ is not a Mahlo cardinal, then there exists an \aleph_2 -Souslin tree.

Theorem (2016)

If GCH holds and $L \models (\aleph_2)^V$ is not weakly compact, then there exists an \aleph_2 -Souslin tree with no \aleph_1 -Aronszajn subtrees.

The \aleph_2 -Souslin problem (cont.)

Theorem (Gregory, 1976)

If GCH holds and $L \models (\aleph_2)^V$ is not a Mahlo cardinal, then there exists an \aleph_2 -Souslin tree.

Theorem (2016)

If GCH holds and $L \models (\aleph_2)^V$ is not weakly compact, then there exists an \aleph_2 -Souslin tree with no \aleph_1 -Aronszajn subtrees.

This is sharp:

Theorem (Todorcevic, 1981)

If $L \models \kappa$ is weakly compact, then for some forcing poset $\mathbb{P} \in L$, $L^{\mathbb{P}} \models$ GCH holds, $\kappa = \aleph_2$, and every \aleph_2 -Aronszajn tree contains an \aleph_1 -Aronszajn subtree.

The general statements

Recall

$\square(\kappa)$ asserts the existence of a particular ladder system over κ .
By a 1987 theorem of Todorćević, if κ is a regular uncountable cardinal and $\square(\kappa)$ fails, then $L \models \kappa$ is weakly compact.

The general statements

Theorem (2016)

For every uncountable cardinal λ , $\square(\lambda^+) + \text{GCH}$ implies:

- ▶ *There exists a λ^+ -Souslin tree;*

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The general statements

Theorem (2016)

For every uncountable cardinal λ , $\square(\lambda^+) + \text{GCH}$ implies:

- ▶ There exists a *club-regressive* λ^+ -Souslin tree;

Remark

A *club-regressive* κ -tree contains no Cantor subtrees nor ν -Aronszajn subtrees for every regular cardinal $\nu < \kappa$.

Recall

$\square(\kappa)$ asserts the existence of a particular ladder system over κ . By a 1987 theorem of Todorćević, if κ is a regular uncountable cardinal and $\square(\kappa)$ fails, then $L \models \kappa$ is weakly compact.

The general statements

Theorem (2016)

For every uncountable cardinal λ , $\square(\lambda^+) + \text{GCH}$ implies:

- ▶ There exists a club-regressive λ^+ -Souslin tree;
- ▶ There exists a $\text{cf}(\lambda)$ -complete λ^+ -Souslin tree.

Remark

A club-regressive κ -tree contains no Cantor subtrees.

θ -complete: any increasing sequence of length $< \theta$ admits a bound.

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For every cardinal $\lambda \geq \beth_\omega$, $\square(\lambda^+) + \text{CH}_\lambda$ implies:

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Remark

CH_λ stands for the local instance of GCH: $2^\lambda = \lambda^+$.

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For every cardinal $\lambda \geq \beth_\omega$, $\square(\lambda^+) + \text{CH}_\lambda$ implies:

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Theorem (Brodsky-Rinot, 201 ∞)

For every singular strong limit cardinal λ , $\square(\lambda^+) + \text{CH}_\lambda$ implies:

- ▶ There exists a *uniformly coherent, prolific* λ^+ -Souslin tree.

The general statements

A consequence of uniformly coherent prolific κ -Souslin tree

There exists a symmetric coloring $c : [\kappa]^2 \rightarrow \kappa$ such that:

- ▶ every $A \subseteq \kappa$ of size κ is omnichromatic, i.e., $c \upharpoonright [A]^2$ is onto κ ;
- ▶ for every $\beta < \gamma < \kappa$, $\{\alpha < \beta \mid c(\alpha, \beta) \neq c(\alpha, \gamma)\}$ is finite.

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The microscopic approach to Souslin-tree construction

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- ▶ Constructions of κ -Souslin trees depend on the nature of κ : successor of regular, successor of singular of countable cofinality, successor of singular of uncountable cofinality, or inaccessible.
- ▶ Constructions involve sealing of antichains at some set $S \subseteq \kappa$ of levels, such that S is a **non-reflecting stationary set**;

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- ▶ Constructions of κ -Souslin trees depend on the nature of κ : successor of regular, successor of singular of countable cofinality, successor of singular of uncountable cofinality, or inaccessible.
- ▶ Constructions involve sealing of antichains at some set $S \subseteq \kappa$ of levels, such that S is a **non-reflecting stationary set**; By a 1985 theorem of Harrington and Shelah, a Mahlo cardinal suffices for the consistency of “every stationary subset of $\{\alpha < \omega_2 \mid \text{cf}(\alpha) = \omega\}$ reflects”, and so in view of the goal of deriving a weakly compact cardinal, there is a need for a construction that does not appeal to non-reflecting stationary sets.

The microscopic approach to Souslin-tree construction

Back in 2009, as a first step towards attacking the higher Souslin problem, I reviewed all known combinatorial constructions of Souslin trees (of various kinds). I found out:

- ▶ Constructions of κ -Souslin trees depend on the nature of κ : successor of regular, successor of singular of countable cofinality, successor of singular of uncountable cofinality, or inaccessible.
- ▶ Constructions involve sealing of antichains at some set $S \subseteq \kappa$ of levels, such that S is a non-reflecting stationary set;
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- ▶ Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another;
- ▶ For inaccessible κ , there is a dearth of combinatorial constructions.

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Find a proxy!

1. Introduce a combinatorial principle from which **many** constructions can be carried out **uniformly**;
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As a proof of concept, I introduced the principle $\diamond_{\lambda, < \mu}^{\Gamma}$.

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Definition

$\diamond_{\lambda, < \mu}^{\Gamma}$ asserts the existence of two sequences, $\langle \mathcal{C}_{\alpha} \mid \alpha < \lambda^+ \rangle$ and $\langle \varphi_{\theta} \mid \theta \in \Gamma \rangle$, such that all of the following holds:

- ▶ $\langle \mathcal{C}_{\alpha} \mid \alpha < \lambda^+ \rangle$ is a $\square_{\lambda, < \mu}$ -sequence;
- ▶ Γ is a non-empty set of regular cardinals $< \lambda^+$;
- ▶ $\varphi_{\theta} : [\lambda^+]^{< \lambda} \rightarrow [\lambda^+]^{\leq \lambda}$ is a function, for all $\theta \in \Gamma$;
- ▶ for every subset $A \subseteq \lambda^+$, every club $D \subseteq \lambda^+$, and every $\theta \in \Gamma$, there exists some $\alpha \in E_{\theta}^{\lambda^+}$ such that for every $C \in \mathcal{C}_{\alpha}$, the following holds:

$$\sup\{\delta \in \text{nacc}(\text{acc}(C)) \cap D \mid \varphi_{\theta}(C \cap \delta) = A \cap \delta\} = \alpha.$$

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For some reason, I was displeased with the above definition, and gave up on this project.

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Some of my friends complained that our proxy principle has 8 parameters. My response:

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A construction á la microscopic approach

```
#include <NormalTree.h>
#include <SealAntichain.h>
#include <SealAutomorphism.h>
//#include <Specialize.h>
//#include <SealProductTree.h>
```

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If \vec{C} happens to be a witness to $P(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$, then the outcome tree $\mathcal{T}(\vec{C})$ will be κ -Souslin. The “better” the vector of parameters (κ, \dots) is, the “better” $\mathcal{T}(\vec{C})$ we get.

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Unlike previous constructions, the microscopic approach lacks any knowledge of its goals, or where it is heading. In particular, **book-keeping, counters, timers, and non-reflecting stationary sets play no role.**

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Unlike previous constructions, the microscopic approach lacks any knowledge of its goals, or where it is heading. In particular, book-keeping, counters, timers, and non-reflecting stationary sets play no role. It is only from the outside that we can analyze \vec{C} and decide its effect on the properties of $\mathcal{T}(\vec{C})$.

The microscopic approach to Souslin-tree construction

The parameterized proxy principle proved to be as flexible as we hoped. We went case by case, and proved that any \diamond -based scenario that was known to yield a κ -Souslin tree already yields an instance of $P(\kappa, \dots)$ from which the existence of a whole gallery of κ -Souslin trees may now be derived.

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And we found new scenarios, and new types of trees.

New scenarios

Open problem (Schimmerling)

Suppose that λ is a singular cardinal.

Does $\text{GCH} + \square_{\lambda}^*$ entail the existence of a λ^+ -Souslin tree?

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Counterexamples around the combinatorics of successor of singulars often make use of forcing notions to singularize a large cardinal (e.g., Prikry/Magidor/Radin forcing).

However, we have identified the following obstruction:

Theorem (Brodsky-Rinot, 2016)

Suppose that λ is a strongly inaccessible cardinal, and CH_{λ} holds. If \mathbb{Q} is a λ^+ -cc notion of forcing of size λ^+ that makes λ into a singular cardinal, then \mathbb{Q} introduces $\text{P}(\lambda^+, \dots)$, so that, in particular, \mathbb{Q} introduces a λ^+ -Souslin tree.

New type of Souslin trees

Theorem (Devlin, 1980's)

Assume $V = L$.

- ▶ *There exists an \aleph_2 -Souslin tree \mathcal{T} and ultrafilter $\mathcal{U} \subseteq [\aleph_0]^{\aleph_0}$, for which the reduced power tree $\mathcal{T}^{\aleph_0}/\mathcal{U}$ is not \aleph_2 -Aronszajn;*
- ▶ *There exists an \aleph_2 -Souslin tree \mathcal{T} and ultrafilter $\mathcal{U} \subseteq [\aleph_0]^{\aleph_0}$, for which the reduced power tree $\mathcal{T}^{\aleph_0}/\mathcal{U}$ is \aleph_2 -Aronszajn.*

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By swallowing the bookkeeping, counters and timers into the proxy principle, one is now able to construct far more complicated trees.

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Then there exist trees $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$,
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	\mathcal{T}	$\mathcal{T}^{\aleph_0}/\mathcal{U}_0$	$\mathcal{T}^{\aleph_1}/\mathcal{U}_1$
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Theorem (Brodsky-Rinot, 2017)

If $V = L$, then there exists an \aleph_6 -Souslin tree \mathcal{T} ,
along with ultrafilters $\mathcal{U}_n \subseteq [\aleph_n]^{\aleph_n}$ for each $n < 6$, such that:

$\mathcal{T}^{\aleph_n}/\mathcal{U}_n$ is \aleph_6 -Aronszajn iff n is not a prime number.

Classification of Aronszajn trees

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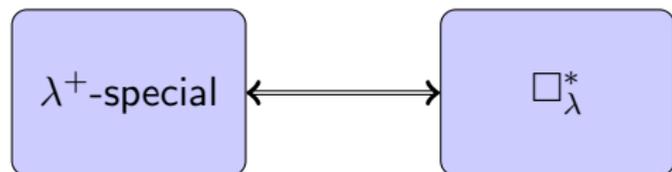
Theorem (Baumgartner-Malitz-Reinhardt, 1970)

It is consistent that all \aleph_1 -Aronszajn trees are special.

Weak squares and (non)special trees

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Suppose that λ is a singular cardinal, and $2^{\lambda} = \lambda^+$.

If λ is a strong limit and \square_{λ}^* holds, then there exists a nonspecial λ^+ -Aronszajn tree.



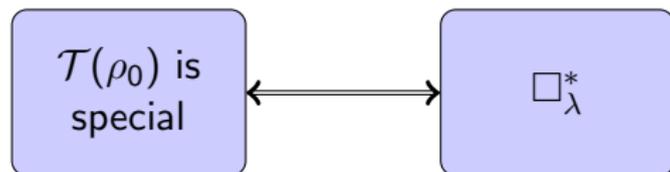
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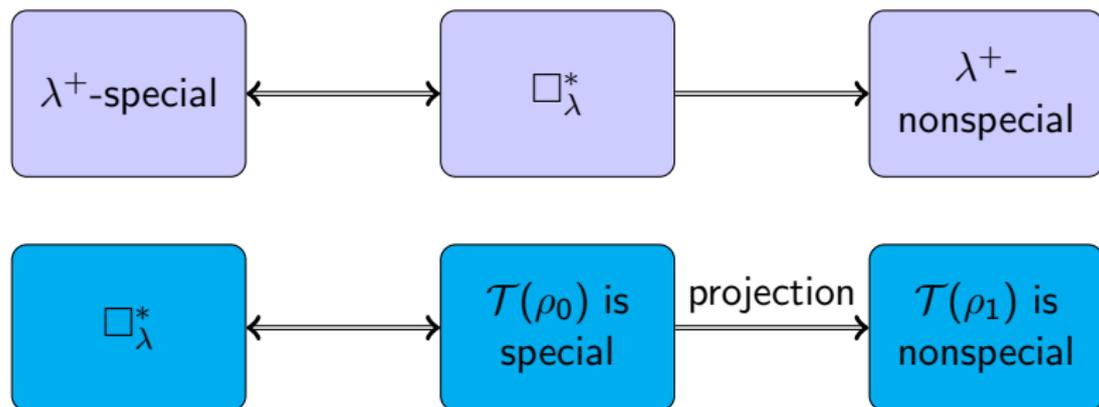
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