ULTRAFILTERS

AND PARTITION RELATIONS

PART 3 (TALK BY ANDREAS BLASS)

IF $\omega \to [\mathcal{U}]^2_4$, THEN $\mathcal{U}$ IS A P-POINT.

SUPPOSE $\mathcal{U}, \mathcal{V}_0, \mathcal{V}_1, \ldots$ ARE PAIRWISE NONISOMORPHIC SELECTIVE ULTRAFILTERS AND

$$\mathcal{W} = \mathcal{U} - \sum_N \mathcal{V}_N,$$

THEN $\mathcal{W}$ IS NOT A P-POINT AND $\omega^2 \to [\mathcal{W}]^2_5$.

THE FACT THAT $\mathcal{W}$ IS NOT A P-POINT IS IMMEDIATE. THE FACT THAT $\omega^2 \to [\mathcal{W}]^2_5$ REQUIRES PROOF.

RECALL: FOR SELECTIVE $\mathcal{U}$, $\omega \to (\mathcal{U})^2_2$. FOR ANY $\mathcal{U}, \mathcal{V}$, IF $[\omega]^2$ IS PARTITIONED INTO FINITELY MANY PIECES, THEN $\exists X \in \mathcal{U}, Y \in \mathcal{V}$, $l : \omega \to \omega$, l IS NOT CONSTANT ON ANY SET IN $\mathcal{V}$ AND ALL PAIRS $\{x, y\}$ WITH $x \in X$, $y \in Y$, AND $x \le l(y)$ ARE IN THE SAME PIECE.

IF $\mathcal{U}, \mathcal{V}$ ARE $\not\cong$, SELECTIVE, THEN THIS WORKS WITH $x < y$.

$$f(\mathcal{U}) = \mathcal{U} \Rightarrow \{x : f(x) = x\} \in \mathcal{U}$$

g IS 1-1 AND $f(\mathcal{U}) = g(\mathcal{U})$

$$\Rightarrow \{x : f(x) = g(x)\} \in \mathcal{U}.$$

PROOF THAT IF $\omega \to [\mathcal{U}]^2_4$, THEN $\mathcal{U}$ IS A P-POINT:

SUPPOSE, TOWARDS A CONTRADICTION, THAT $\mathcal{U}$ IS NOT A P-POINT. LET $p : \omega \to \omega$ BE NEITHER CONSTANT NOR FINITE-TO-ONE ON ANY SET IN $\mathcal{U}$.

PARTITION $[\omega]^2$ INTO THE SETS

1. $\{\{x < y\} : p(y) < p(x) < x < y\}$,
2. $\{\{x < y\} : p(x) = p(y) < x < y\}$,
3. $\{\{x < y\} : p(x) < p(y) < x < y\}$,

④ $\{\{x<y\} : p(x) < x = p(y) < y\}$,

or ⑤ $\{\{x<y\} : p(x) < x < p(y) < y\}$.

NOTE: CAN AVOID THE SET

④ $\{\{x<y\} : p(x) < x = p(y) < y\}$.

THE OTHER 4 MEET $[H]^2$ FOR ALL $H \in \mathcal{U}$.

①: CHOOSE $n \in \omega$ S.T. $p^{-1}(n) \cap H$ IS INFINITE.

CHOOSE $m > n$ S.T. $p^{-1}(m) \cap H$ IS INFINITE.

CHOOSE $x \in p^{-1}(m) \cap H$.

CHOOSE $y \in p^{-1}(n) \cap H$, $y > x$.

②: EASIER THAN ①.

③: CHOOSE m AND THEN $n > m$.

⑤: EASIER. ∎

CLAIM: $\mathcal{W}$ IS A Q-POINT

(EVERY FINITE-TO-ONE MAP $\omega^2 \to \omega$ IS 1-1 ONE A SET IN $\mathcal{W}$)

PROOF OF CLAIM: FOR EACH $n \in \omega$, $f_n : \omega \to \omega$ $(k \mapsto f(n,k))$ IS 1-1 ON A SET $X_n \in \mathcal{V}_n$.

$$\bigcup_{n \in \omega} (\{n\} \times X_n) \in \mathcal{W}.$$

THE ULTRAFILTERS $f_n(\mathcal{V}_n)$ ARE DISTINCT P-POINTS. SO EACH ONE CONTAINS A SET $Y_n \in f_n(\mathcal{V}_n)$ THAT IS IN NONE OF THE OTHERS.

$Y_n \notin f_m(\mathcal{V}_m)$ FOR $m \neq n$.

CAN ARRANGE TO HAVE Y_n'S PAIRWISE DISJOINT: $Y_n' = Y_n \cap \bigcap_{m<n} (\omega - Y_m) \in f_n(\mathcal{V}_n)$.

THEN $\bigcup_n (\{n\} \times (f_n^{-1}(Y_n') \cap X_n)) \in \mathcal{W}$

AND f IS 1-1 ON IT. ∎

LET $p : \omega^2 \to \omega$ BE THE FUNCTION $(n,m) \mapsto n$

AND $q : \omega^2 \to \omega$ BE THE FUNCTION $(n,m) \mapsto m$.

① $\{\{x<y\} : p(y) < p(x) < q(x) < q(y)\}$

② $\{\{x<y\} : p(y) = p(x) < q(x) < q(y)\}$

③ $\{\{x<y\} : p(x) < p(y) < q(x) < q(y)\}$

⓪ $\{\{x<y\} : p(x) < p(y) = q(x) < q(y)\}$

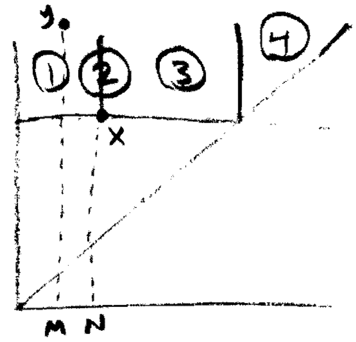
④ $\{\{x<y\} : p(x) < q(x) < p(y) < q(y)\}$, GIVEN ANY PARTITION of $[\omega^2]^2$ INTO FINITELY MANY PIECES, THERE IS $H \in \mathcal{W}$ S.T. ALL PAIRS $\in [H]^2$ AS IN ① ARE IN ONE PIECE,

" " ② " ",

" " ③ " ",

" " ④ " ", AND

NO PIECES $\in [H]^2$ SATISFY ⓪.



② : FOR EACH $N \in \omega$, THERE IS $X_N \in \mathcal{V}_N$ S.T. ALL $k < l$ IN X_N HAVE THE SAME COLOR c_N FOR $\{(N,k), (N,l)\}$. THERE IS $Y \in \mathcal{U}$ S.T. c_N IS THE SAME c FOR ALL $N \in Y$. THEN

$$\bigcup_{N \in Y} (\{N\} \times X_N) \in \mathcal{W}$$

WORKS FOR ②.

① : TEMPORARILY FIX $M < N$ IN ω. AS $\mathcal{V}_M, \mathcal{V}_N$ ARE $\not\equiv$ AND SELECTIVE, GET $X_{N,M} \in \mathcal{V}_N$, $Y_{M,N} \in \mathcal{V}_M$ S.T. ALL PAIRS $\hat{x} \in X_{N,M}$, $\hat{y} \in Y_{M,N}$ WITH $\hat{x} < \hat{y}$ HAVE THE SAME COLOR $c_{M,N}$ FOR

$$\{\underbrace{(N,\hat{x})}_{x}, \underbrace{(M,\hat{y})}_{y}\}.$$

W.L.O.G. THE SAME $c_{M,N}$ FOR ALL M,N (BY RESTRICTING TO A SET IN $\mathcal{U}$).

LET $X_N = \bigcap_{M<N} X_{N,M}$.

THIS $\in \mathcal{V}_N$. TEMPORILY FIX M.
PARTITION PAIRS $\{N < \hat{y}\}$ ACCORDING TO WHETHER $\hat{y} \in Y_{M,N}$.
FIND $Y_M \in \mathcal{V}_M$ AND $Z \in \mathcal{U}$ S.T. ALL PAIRS $\{N < \hat{y}\}$ WITH $N \in Z$, $\hat{y} \in Y_M$ ARE IN ONE PIECE.

FOR FIRST $N \in Z$, Y_M MEETS $Y_{M,N}$.

$\in \mathcal{V}_M$

$\therefore \forall \hat{y} \in Y_M \; \forall_{N \in Z}$

$N < \hat{y} \Rightarrow \hat{y} \in Y_M$.