ULTRAFILTERS
AND PARTITION RELATIONS
PART 4

(TALK BY ANDREAS BLASS)

SELECTIVE, $\not\cong$

$$\mathcal{U}\text{-}\sum_N \mathcal{V}_N = \mathcal{W}$$

$\omega^2 \longrightarrow [\mathcal{W}]^2_5 \qquad F:[\omega^2]^2 \to 5$

$p(\mathcal{W}) = \mathcal{U}$

$q(\mathcal{W}) \cong \mathcal{W}$ ($\mathcal{W}$ IS A Q-POINT)

$$p(x) < q(x) < q(y)$$

WE CAN CLASSIFY PAIRS (x,y) BASED ON WHERE $p(y)$ IS:

① $p(y) < p(x) < q(x) < q(y)$

② $p(y) = p(x) < q(x) < q(y)$

③ $p(x) < p(y) < q(x) < q(y)$

⓪ $p(x) < p(y) = q(x) < q(y)$

④ $p(x) < q(x) < p(y) < q(y)$.

CASE 0 IS NOT POSSIBLE.

LAST TIME, WE DELT WITH ① AND ②.

TEMPORILY FIX $m < n \in \omega$.

THERE ARE $X \in \mathcal{V}_{m,n}$, $Y \in \mathcal{V}_{n,m}$ S.T. ALL PAIRS $\{\langle m,\hat{x}\rangle, \langle n,\hat{y}\rangle\}$ WITH $\hat{x} \in X_{m,n}$, $\hat{y} \in Y_{n,m}$, $\hat{x} < \hat{y}$

HAVE SAME COLOR $c_{m,n}$.

UNFIX m,n. THERE IS $Z \in \mathcal{U}$ S.T. $c_{m,n}$ IS THE SAME FOR ALL $m < n$ IN Z.

LET $Y_n = \bigcap_{m<n} Y_{n,m}$ SO $Y_n \in \mathcal{V}_n$.

TEMPORARILY FIX m.

PARTITION $[\omega]^2$ BY

$$\underset{n<\hat{x}}{\{n,\hat{x}\}} \longmapsto \begin{cases} 0 & \text{if } \hat{x} \in X_{m,n} \\ 1 & \text{OTHERWISE.} \end{cases}$$

GET $W_m \in \mathcal{U}$, $V_m \in \mathcal{V}_m$ S.T.

$n \in W_m$, $\hat{x} \in V_m$, $n < \hat{x} \Longrightarrow$ SAME 0 ~~or 1~~

(PICK $n \in W$. PICK $\hat{x} > n$, $\hat{x} \in V \cap X_{m,n}$. $\{n,\hat{x}\}$ HAS COLOR 0, $\therefore$ ALL PAIRS HAVE COLOR 0.)

RESTRICT TO POINTS x,y WITH $p(x) \in Z$, $q(x) \in Y_{p(x)} \cap V_{p(x)}$.

If $m < n$ AND $n \in W_m$, THEN

$$F\{\langle m,\hat{x}\rangle, \langle n,\hat{y}\rangle\} = c.$$

By THE RAMSEY PROPERTY OF $\mathcal{U}$, GET $U \in \mathcal{U}$ S.T.

ALL ~~OR NONE~~ OF THE PAIRS $\{M < N\} \in [U]^2$ HAVE $N \in W_M$.

$\left(\text{PICK } M \in U. \text{ PICK } \underset{M}{\overset{\vee}{N}} \in U \cap W_M \quad \underset{\text{GOOD}}{\{M < N\}} \right)$.

THUS, WE HAVE DEALT WITH CASE 3.

THE ONE REMAINING CASE IS 4.

RECALL: $(\exists X \in \mathcal{W})$

$(\exists l : \omega^2 \to \omega)$ S.T.

- l IS NOT CONSTANT ON ANY SET $\in \mathcal{W}$, AND
- F IS CONSTANT ON ALL PAIRS $\{x, y\} \in [X]^2$ WITH $q(x) < l(y)$.

IF $p \leq l$ ON SOME SET $\in \mathcal{W}$, THEN DONE.

$\therefore$ W.L.O.G., $l < p$ ON A SET $\in \mathcal{W}$.

l TAKES ONLY FINITELY MANY VALUES ON ANY $\{M\} \times \omega$. l IS CONSTANT ON A SET $Y_M \in \mathcal{V}_M$. RESTRICTING TO A SET $\in \mathcal{W}$, GET

$l = g \circ p$ FOR SOME $g : \omega \to \omega$.

$g(M) < M$.

WANT TO GET FROM $q(x) < p(y)$ TO $q(x) < (g \circ p)(y)$.

g NOT CONST ON ANY SET $\in \mathcal{U}$.

g IS 1-1 ON A SET $\in \mathcal{U}$.

PARTITION ω INTO FINITE INTERVALS $I_0, I_1, \ldots$ S.T. $(\forall_{x \in I_{N+1}})$ $g(x) \in I_N \cup I_{N+1}$. W.L.O.G. $\mathcal{U} \ni \bigcup_{k \in \omega} I_{3k+2}$.

DEFINE $h : \omega \to \omega$:

ON I_N, h TAKES $\lfloor \frac{N}{3} \rfloor$.

WANT $Z \in \mathcal{W}$ S.T. FOR $x, y \in Z$, $h(p(y)) \neq h(q(x))$.

SUFFICES TO FIND R WITH

$(h \circ p)^{-1}(R) \in \mathcal{W}$ AND

$(h \circ q)^{-1}(\omega - R) \in \mathcal{W}$.

$R \in h(p(\mathcal{W})) = h(\mathcal{U})$.

$\omega - R \in h(q(\mathcal{W}))$.

SUFFICES TO SHOW $h(\mathcal{U}) \neq h(q(\mathcal{W}))$.

h IS A FINITE-TO-ONE FUNCTION.

WE HAVE $\mathcal{U} \cong h(\mathcal{U})$ AND $\mathcal{W} \cong h(q(\mathcal{W}))$. THIS PROVES THE RESULT. ∎