Some Consequences of I_0 in Higher Degree Theory

Xianghui Shi
Beijing Normal University

Workshop on Iterated Forcing and Large Cardinals
Fields Institute, November 15, 2012

This is a joint work with W.Hugh Woodin.

Dedicated to Professor Richard Laver.

Main results
- Axiom I_0
- Large Perfect Set Theorem
- Posner-Robinson Theorem
- Degree Determinacy

Higher Degree Theory
- Degree theoretic questions
- In L, $L[\mu]$
- In $L[\bar{\mu}]$ and beyond
- Degree structures under I_0

Failure of Degree Determinacy
- Preparation
- Proof of main lemma, sketch

Axiom I_0

Motivation

- Part of I_0 theory.
- Part of higher degree theory.

Definition

I_3. $\exists j : V_\lambda \to V_\lambda$ $\qquad \mathcal{E}(V_\lambda) \neq \varnothing$

I_1. $\exists j : V_{\lambda+1} \to V_{\lambda+1}$ $\qquad \mathcal{E}(V_{\lambda+1}) \neq \varnothing$

I_0. $\exists j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ $\qquad \mathcal{E}(L(V_{\lambda+1})) \neq \varnothing$

Here j stands for a nontrivial elem embedding with $\mathrm{crit}(j) < \lambda$.
The $\mathcal{E}$ inequalities on the right are Laver's notation.

- $I_0 \Rightarrow I_1 \Rightarrow I_3$.
- By Kunen, ZFC $\Rightarrow \mathcal{E}(V_{\lambda+2}) = \varnothing$. These are the strongest large cardinals not known to be inconsistent with ZFC.
- There is a strong resemblance between structural properties of subsets of $V_{\lambda+1}$ under ZFC $+ I_0$ and those of subsets of $\mathbb{R} = V_{\omega+1}$ under ZF $+$ DC $+$ AD.

$$\frac{\mathsf{AD}}{L(\mathbb{R})} \sim \frac{I_0}{L(V_{\lambda+1})}$$

  We add two more instances that re-affirm this analogy.
- The analogy is not perfect. Our last result is an evidence in this direction.

Large Perfect Set Theorem

Theorem 1 (Large Perfect Set Theorem)

Assume I_0. Then every subsets of $V_{\lambda+1}$ that is definable over $(V_{\lambda+1}, \in)$ has Large Perfect Set Property.

- The topology on $V_{\lambda+1}$ is given by the basic open sets $O_{a,\alpha} = \{b \subset V_\lambda \mid b \cap V_\alpha = a\}$, $\alpha < \lambda$, $a \subset V_\alpha$.
- Let $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ be the critical sequence: $\kappa_0 = \mathrm{crit}(j)$, and $\kappa_{n+1} = j(\kappa_n)$. Identify $V_{\lambda+1}$ as $|V_{\kappa_0}| \times \prod_i |V_{\kappa_{i+1}} - V_{\kappa_i}|$.
- For any $\bar{\lambda} = \langle \lambda_i : i < \omega \rangle$ with $\sup \lambda_i = \lambda$, a $\bar{\lambda}$-splitting tree is a subtree of the full tree that is isomorphic to $S_{\bar{\lambda}} = (\bigcup_n (\prod_{i<n} \lambda_i), \sqsubseteq)$.
- $X \subseteq V_{\lambda+1}$ has LPS[1] Property if either $|X| \leq \lambda$ or $X \supseteq [T]$, where T is $\bar{\lambda}$-splitting, for some $\bar{\lambda}$ with $\sup \lambda_i = \lambda$.

[1] LPS = Large Perfect Set

- This is a "projective" version.
- One can improve it to sets in $L_\lambda(V_{\lambda+1})$, using the machinery of $\mathbb{U}(j)$-representable sets developed in Woodin's *Suitable Extender Model, II*.
- Cramer (2012) improves it to all sets in $L(V_{\lambda+1})$, using the technique of inverse limit reflection.

In the context of AD and $L(\mathbb{R})$,

- (Davis, 64). Every set of reals in $L(\mathbb{R})$ has PSP.
- (Sami, 95). This also follows from Turing Determinacy (TD).

Posner-Robinson Theorem

Fix a well-ordering $w : H(\lambda) \to \lambda$, a reasonable fragment $\Gamma \subsetneq \mathrm{ZFC}$. For $a, b \subset \lambda$:

- $M[a]$ denotes the minimal Γ-model of the form $L_\alpha[w, a]$, $\alpha > \lambda$. Let α_a, Γ-ordinal for a, denote the height of $M[a]$.
- $a \leq_\Gamma b$ if $M[a] \subseteq M[b]$. $a \equiv_\Gamma b$ if $a \leq_\Gamma b$ and $b \leq_\Gamma a$
- Write $\underset{\sim}{a}$ for the degree of a, the $\equiv_\Gamma$-equivalence class of a.
- $J_\Gamma(a)$, Γ-jump of G, is the theory of $M[a]$. It can be coded by a subset of λ.

The following is a corollary of LPS Theorem.

Theorem 2 (Posner-Robinson Theorem at λ)

Assume I_0. Then for almost all (λ many exceptions) $X \subset \lambda$,

$$(\exists G \subset \lambda\ )\, [(X, G) \equiv_\Gamma J_\Gamma(G)].^2$$

[2] True for finer equivalence as well, e.g. $(x, G) \equiv_{\Sigma^0_1(V_\lambda)} G^\sharp$.

Classical Posner-Robinson

1. (Posner-Robinson, etc.) If $x \subset \omega$ and $x \notin \Delta^0_1$, then
$$(\exists G)[(x, G) \equiv_T G'].$$
2. (Shore-Slaman) If $x \in \mathscr{P}(\omega) \setminus L_\alpha$, $\alpha < \omega_1^{\mathrm{CK}}$, then
$$(\exists G)[(x, G) \equiv_T G^{(\alpha)}].$$
3. (Woodin) If $x \in \mathscr{P}(\omega) \setminus L_{\omega_1^{\mathrm{CK}}}$, then $(\exists G)[(x, G) \equiv_T O^G]$.
4. (Woodin) If $x \in \mathscr{P}(\omega) \setminus L$, then $(\exists G)[(x, G) \equiv_T G^\sharp]$.

Slaman-Steel (early 80's) used 2. in their (partial) solution to Martin Conjecture:

(ZF + DC + AD). Degree inv. functions on $\mathbb{R}$ are pre-wellordered by $f \leq_m g$ iff $f(x) \leq_T g(x)$ on a cone. Let f' be s.t. $\underset{\sim}{f'(x)} = \underset{\sim}{x}'$.

$$\mathrm{rank}_{\leq_m}(f) = \alpha \quad \Rightarrow \quad \mathrm{rank}_{\leq_m}(f') = \alpha + 1.$$

Degree Determinacy

For the talk, we fix $\Gamma = \mathsf{Z}$, Zermelo Set Theory.

- A set $A \subset \mathscr{P}(\lambda)$ is Z-degree invariant if $a \in A \Rightarrow \underset{\sim}{a} \subset A$.
- A cone is a set of the form $C_a = \{b \mid a \leq_{\mathsf{Z}} b\}$.
- $\mathrm{Det}_\lambda(\text{Z-Deg})$: Every Z-degree invariant subset of $\mathscr{P}(\lambda)$ either contains a cone or is disjoint from a cone.

Theorem 3 (ZFC)

Assume $j \in \mathcal{E}(L(V_{\lambda+1}))$ and in V_λ, $\kappa_0 = \mathrm{crit}(j)$ is supercompact, and it supercompactness is indestructible by κ_0-directed posets.

$$L(V_{\lambda+1}) \models \neg\mathrm{Det}_\lambda(\text{Z-Deg}).$$

Denote the hypothesis as ${I_0}^*$.

Outline of the proof

We sketch the idea of the proof, modulo main technical lemma.

- Strategy: Show that $\mathrm{Det}_\lambda(\text{Z-}\mathrm{Deg})$ implies the existence of *ω_1-sequence of distinct reals*.

Lemma (Kechris-Kleinberg-Moschovakis-W, Woodin)

Suppose there is a countably additive measure μ on $[\lambda^+]^{\omega_1}$ that satisfies the following coherence condition: $\forall A \subset [\lambda^+]^{\omega_1}$, $\forall P \subset \omega_1$ with $\mathrm{otp}(P) = \omega_1$,

$$\mu(A) = 1 \quad \Rightarrow \quad \mu(A|P) = 1,$$

where $A|P =_{\text{def}} \{a \restriction P \mid a \in A\}$. Then every ω_1-Suslin set is determined.

- The point is to produce such a partition measure on $[\lambda^+]^{\omega_1}$.

- The following lemma provides the means for transferring the cone measure on $\mathscr{P}(\omega)$ to a partition measure on $[\omega_1]^\omega$.

Lemma (Jensen)

Suppose $A = \{\alpha_i : i < \omega\}$ is a set of a-admissible ordinals, $a \subset \omega$. And $\mathrm{otp}(A) = \omega$. Then $\exists b \geq_T a$ s.t.

$$A = \text{first } \omega \text{ many } b\text{-admissible ordinals.}$$

- Martin used this to show that AD $\Rightarrow \omega_1 \to (\omega_1)^\omega$.
- A coherent system of measures were used to prove AD from infinite exponent partition relations.
- The singularity of λ presents an obstacle for a direct generalization of Jensen's lemma. (for $\mathrm{otp}(A) > \mathrm{cf}(\lambda)$)
- Moreover, $\mathrm{cf}(\lambda) = \omega$ seems to prevent us from getting a ω_1-exponent partition measure for $[\lambda^+]^{\omega_1}$. (Indestructibility comes in)

- For $a \subset \lambda$, $Z_a =_{\text{def}} \{\alpha_i \mid \alpha_i > \lambda \text{ is the } i\text{-th Z-ordinal}, i < \omega_1\}$
- Define μ on $[\lambda^+]^{\omega_1}$ as follows: for $A \subset [\lambda^+]^{\omega_1}$,

  $$\mu(A) = 1 \quad \text{iff} \quad A \supseteq \mathfrak{C}_a =_{\text{def}} \{Z_b \mid b \geq_\Gamma a\}, \text{ for some } a.$$
- Next lemma helps to get around the obstacle and to obtain the *Coherence* condition, but with a price of an additional assumption.

Main Lemma

Assume ZFC $+ {I_0}^* + \mathrm{Det}_\lambda(\text{Z-}\mathrm{Deg})$. Then $\forall u \subset \lambda$, $\forall P \subset \omega_1$,

$$\exists a, b \geq_{\mathsf{Z}} u \text{ s.t. } Z_b = Z_a \restriction P.$$

- μ is a countably additive and *coherent* measure on $[\lambda^+]^{\omega_1}$.

Q.E.D.

A Conjecture

We just argued that under ${I_0}^*$, Det_λ(Z-Deg) fails. In fact, we make the following conjecture.

Conjecture (ZFC)

$L(\mathscr{P}(\lambda)) \models \neg\mathrm{Det}_\lambda$(Z-Deg), for any uncountable cardinal λ.

Here are some evidence:

Case 1. λ is strong limit and $\mathrm{cf}(\lambda) > \omega$.

Theorem (Shelah) (ZFC)

If λ is strongly limit and $\mathrm{cf}(\lambda) > \omega$, then $L(\mathscr{P}(\lambda)) \models$ AC.

AC can give us two disjoint sequences of cofinal degrees. Thus Det_λ(Z-Deg) is false in $L(\mathscr{P}(\lambda))$.

Case 2. λ regular.

- λ is regular and $2^{<\lambda} = \lambda$.

  Suppose NOT. Jensen's lemma can be generalized to regular cardinals that satisfy $2^{<\lambda} = \lambda$, and so there is a coherent partition measure on $[\lambda^+]^{\omega_1}$. But in $L(\mathscr{P}(\lambda))$, $\mathbb{R}$ is well-ordered. Contradiction!

- λ is regular.

  If $L(\mathscr{P}(\lambda)) \models \mathrm{Det}_\lambda(\text{Z-Deg})$, then $\exists a \subset \lambda$, in fact, a cone of a, s.t. $L[a] \models$ "$L(\mathscr{P}(\lambda)) \models \mathrm{Det}_\lambda(\text{Z-Deg})$". But $2^{<\lambda} = \lambda$ holds in $L[a]$, if λ is regular. Contradiction!

So either case, $\mathrm{Det}_\lambda(\text{Z-Deg})$ is false in $L(\mathscr{P}(\lambda))$.

Case 3. λ is not a strong limit. Unknown.

Next we shall look into degree structures in inner models, which suggests that it is going to be subtle to resolve this conjecture.

Higher Degree Theory

- Higher Degree Theory
  - studies definability degree structures at uncountable cardinals,
  - focus on the connection between large cardinals and degree structures.
- α-recursion theory (for $\alpha > \omega$) is part of higher degree theory. But early studies mostly concern degrees within L, and involves no large cardinals.
- Recent developments reveal some deep connection between large cardinals and degree structures at uncountable cardinals, in particular, strong limit singular of countable cofinality.
- This is a new line of research. Consequences of I_0 presented in this talk are evidences for this connection from one extreme.

A list of questions

- Shall study degree structures in some canonical inner models.
  - Unlike the situation of ω, not very much of degree structures at uncountable cardinals can be determined by ZFC alone.
  - Fine structure models provide more complete settings.
- One can explore various degree notions, in this talk we focus on Z-degrees. The point is that Zermelo set theory is enough for proving Covering.
- A list of degree theoretic questions.
  1. (Post Problem). Are there incomparable degrees, i.e.
  $$\neg(\underset{\sim}{a} \leq \underset{\sim}{b}) \wedge \neg(\underset{\sim}{b} \leq \underset{\sim}{a})?$$
  2. (Minimal Degree). Given $\underset{\sim}{a}$, is there a $\underset{\sim}{b}$ minimal above $\underset{\sim}{a}$, i.e.
  $$\underset{\sim}{a} < \underset{\sim}{b} \wedge \neg\exists\underset{\sim}{c}(\underset{\sim}{a} < \underset{\sim}{c} < \underset{\sim}{b})?$$
  3. (Posner-Robinson). Is it true for co-λ many $x \subset \lambda$ that
  $$(\exists G)[(x, G) \equiv_{\mathsf{Z}} J_{\mathsf{Z}}(G)]?$$
  4. (Degree Determinacy). Is $\mathrm{Det}_{\lambda}(\text{Z-Deg})$ true?

- $\mathrm{cf}(\lambda) = \lambda$. *Not very interesting.*

  Most degree theoretic constructions at ω can be generalized to strongly inaccessible cardinals.

- $\mathrm{cf}(\lambda) > \omega$. *Nothing interesting left.*

Theorem (Sy Friedman, 81) ($V = L$)

The analog of Turing degrees at singular cardinals of uncountable cofinality are well-ordered above a singularizing degree.

The key to this is the analysis of *stationary subsets* of $\mathrm{cf}(\lambda)$.

Corollary ($V =$ any fine structure model)

Z-degrees at singular cardinals of uncountable cofinality are well-ordered above a singularizing degree.

- $\mathrm{cf}(\lambda) = \omega$. *Where the fun is.*

Pictures in L

Observation. $(V = L)$

If $\mathrm{cf}(\lambda) = \omega$, then Z-degrees at λ are well-ordered above a singularizing degree. In particular, Z-degrees at $\aleph_\omega$ is well-ordered.

Proof.

- Suppose $a \subset \lambda$, $a \geq_{\mathsf{Z}} b$, and b singularizes λ. Then a computes a "cutoff" function. Work in $M[a]$. Every $x \subset \lambda$ is identified as a member of $[\lambda]^\omega$.
- $M[a]$ has no sharps, by Covering, $\exists b \in L[w]^{M[a]} \cap \mathscr{P}(\lambda)$ s.t. $a \subset b \wedge |b| \leq \omega_1$. Then
$$\frac{a}{b} \sim \frac{z}{\omega_1}, \text{ for some } z \subset \omega_1.$$
- $M[a]$ and $L[w]^{M[a]}$ have the same $\mathscr{P}(\omega_1)$. Thus $a \in L[w]^{M[a]}$. In other word, $M[a] = L_{\alpha_a}[w]$.
- Γ-degrees at λ are well-ordered above $\underset{\sim}{d}$. $\dashv$

Answers to the list. (above the singularizing degree)

Post Problem	No.
Minimal Degree	Yes. "No" for > 1 minimal covers.
Posner-Robinson	No. Fail to have solution at the limit.
Degree Determinacy	No.

Remark.

- A bit unusual: using Covering *within* L.
- As for inner models between L and $L[\mu]$, such as $L(0^\sharp)$, the same argument applies, since their Covering Lemmas are of the same form.
- A little wrinkle in $L[\mu]$, but still the same picture.

Pictures in $L[\mu]$

Let κ be the measurable, λ strong limit and $\mathrm{cf}(\lambda) = \omega$.

Reorganize $L[\mu]$ as $L[E]$, by Steel's construction, using partial measures. The point is the acceptability condition, i.e. $\forall\gamma < \alpha$,

$$(L_{\alpha+1}[E] - L_\alpha[E]) \cap \mathscr{P}(\gamma) \neq \varnothing \quad \Rightarrow \quad L_\alpha[E] \models |\alpha| = \gamma.$$

Two cases:

- $\lambda > \kappa$. Argue as in L.
- $\lambda < \kappa$. Fix $a \subset \lambda$ above the least $L_{\alpha+1}[E]$ that singularizes λ. $M[a]$ contains no $0^\dagger$. The most $K^{M[a]}$, the core model for $M[a]$, could be is either $L[\mu']$ or there is no measurable.
  - If no measurable, then $M[a] = K^{M[a]}$, by Covering as before. By Comparison, $M[a] \trianglelefteq K = L[\mathcal{E}]$.
  - If $K^{M[a]} = L[\mu']$, then there are two cases.

Covering Lemma for $L[\mu]$. (Dodd-Jensen, 82)

Assume $\neg\exists 0^\dagger$, but there is an inner model $L[\mu]$. Let $\kappa = \mathrm{crit}(\mu)$. Then for every set $x \subset$ Ord, one of the following holds:

1. Every set $x \subset$ Ord is covered by a $y \in L[\mu]$, with $|y| = |x| + \omega_1$.
2. $\exists C$, Prikry generic over $L[\mu]$, s.t. every set $x \subset$ Ord is covered by a $y \in L[\mu][C]$, with $|y| = |x| + \omega_1$.
   Such C is unique up to finite difference.

Case 1. $M[a] \models V = L[\mu']$, as before.

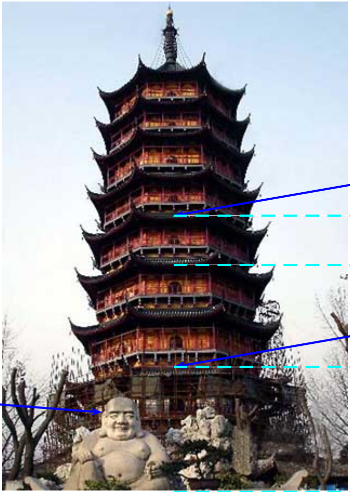
Case 2. Note that $\lambda < \kappa' = \mathrm{crit}(\mu')$, and $C \subset \kappa'$ adds no new bounded subsets of κ'. Some $y \in L[\mu'] \cap \mathscr{P}(\lambda)$ covers a. So $M[a] \models V = L[\mu']$ again.

By Comparison, $M[a] \trianglelefteq K = L[E]$.

Pictures in $L[\bar{\mu}]$

Consider $L[\bar{\mu}]$, $\bar{\mu} = \langle \mu_i : i < \omega \rangle$ is a sequence of measures, and $\kappa_n = \text{crit}(\mu_n)$. The case $\lambda \neq \sup_n \kappa_n$ can be argued as in $L[\mu]$. The Γ-degrees at $\lambda = \sup_n \kappa_n$ present a new structure.

- C in Case 2 of the Covering for $L[\bar{\mu}]$ can be chosen to be an ω-sequence, essentially a diagonal Prikry sequence for $L[\bar{\mu}]$.
- Fix $a \subset \lambda$. $K^{M[a]}$, by Covering, is either of the form $L_\eta[\bar{\mu}]$ or $L_\eta[\bar{\mu}][C_a]$. C_a is Prikry, so $L_\eta[\bar{\mu}]$ in Case 2 is a Z-model. So Z-degrees at λ is pre-wellordered by the associated Z-ordinals.
- Let α_0 be the least Z-ordinal past λ. Note that $a \equiv_{\mathsf{Z}} C_a$, if $\alpha_a = \alpha_0$. Z-degrees associated to α_0 are exactly the ones induced by $\mathcal{C}_0 = \{$diagonal Prikry sequences for $L_{\alpha_0}[\bar{\mu}]\}$.
- Let α_η, $\eta > 0$, be the η-th Z-ordinal above λ. Let $a_{<\eta} \subset \lambda$ codes $\langle \alpha_i : i < \eta \rangle$. Some of $\mathcal{C}_0$ remain to be $L_{\alpha_\eta}[\bar{\mu}]$-generic. Thus the Z-degrees associated to α_η are the ones induced by $a_{<\eta} \oplus \mathcal{C}_0 =_{\mathsf{def}} \{(a_{<\eta}, C) \mid C \in \mathcal{C}_0\}$.



α_η

$\mathrm{Deg}[\mathcal{C}_\eta] = \mathrm{Deg}[a_{<\eta} \oplus \mathcal{C}_0]$

α_0

singularizing degree :-)

$\mathrm{Deg}[\mathcal{C}_0]$.

Answers to the list. (at $\lambda = \sup_n \kappa_n$)

Post Problem	Yes. $\exists$ a LPS[3] of pairwise incomp. degrees.
Minimal Degree	No.
Posner-Robinson	No. Fail to have solution at the limit.
Degree Determinacy	No.

Moreover, there are infinite descending chains of degrees.

Meta-Conjecture

In any reasonable inner model, at every singular λ, $\mathrm{cf}(\lambda) = \omega$, below the least measurable, the Z-degrees are well ordered above some degree.

[3]Reminder: LPS = Large Perfect Set

Picture in $L[\mathcal{U}]$ for $o(\kappa) = \kappa$

Theorem (Yang, 2011)

Assume $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ is a sequence of measurable cardinals s.t. each κ_{n+1} carries κ_n many normal measures. Let $\lambda = \sup_n \kappa_n$. Then there is a minimal Γ-degree above $\underset{\sim}{D}$, where $D \subset \lambda$ codes relevant information, in particular, the above system of measures.

- This can be relativized to degrees above $\underset{\sim}{D}$.
- Yang's argument can produce a large perfect set of minimal degrees, which are automatically pairwise incomparable.
- This picture appears in Mitchell's model for $o(\kappa) = \kappa$.[4]
  - "YES" to Post and Minimal degree questions at $\sup_n \kappa_n$.
  - However, the system of indiscernibles for this inner model is very difficult to analyze.
  - We conjecture "NO" to the other two questions.

[4] Not minimal, but it is the "shortest" with o-expression.

Picture from I_0

Assume $j \in \mathcal{E}(L(V_{\lambda+1}))$. Then

- λ is an ω-limit of measurable cardinals
- λ also satisfies the condition in Yang's Theorem

So answers to the list are as follows:

Post Problem	Yes. $\exists$ a LPS of pairwise incomp. degrees.
Minimal Degree	Yes. $\exists$ a LPS of minimal covers.
Posner-Robinson	Yes.
Degree Determinacy	very likely No.

Remarks

- The complexity of degree structures at certain cardinal reflects the strength of large cardinals in the model.
- Among (fine structure) inner models, the "richness" of the degree structures seem to be correlated to where λ is in the inner model, rather than to the level of the inner model.
- The basic method of using the complexity of the degree structures to get a partition-like property from the degree determinacy can't work in general.
- This means that the proof of the conjecture, i.e.
$$L(\mathscr{P}(\lambda)) \models \neg \mathrm{Det}_\lambda(\textsf{Z-Deg}),$$
from ZFC is going to be subtle.[5]

[5] In inner models, one can proof the conjecture by other means.

Failure of $\mathrm{Det}_\lambda(\Gamma\text{-deg})$

Preparation

Now we prove the technical lemma for the proof of $\neg\mathrm{Det}_\lambda(\text{Z-Deg})$. The power we need from I_0 is the following result in SEM, II.

Theorem (Generic Absoluteness)

Suppose that $j \in \mathcal{E}(L(V_{\lambda+1}))$ is proper and $(M_\omega, j_{0,\omega})$ is the ω-iterate of $(V_\lambda, j \restriction V_\lambda)$. Suppose that $M_\omega[G]$ is a generic extension of M_ω s.t. $G \in V$ and $M_\omega[G] \models \mathrm{cf}(\lambda) = \omega$. Then

$$M_\omega[G] \cap V_{\lambda+1} \prec V_{\lambda+1}.$$

- We omit the definition of properness. The point is that every $j \in \mathcal{E}(L(V_{\lambda+1}))$ can be factored as $j = j_0 \circ k$, where $j_0 \in \mathcal{E}(L(V_{\lambda+1}))$ and is proper.
- If $k \in \mathcal{E}(V_{\lambda+1})$, then $k \restriction V_\lambda \in \mathcal{E}(V_\lambda)$ and is iterable.

By Generic Absoluteness, to prove $V_{\lambda+1} \models \forall u \exists v \varphi$, just force over M_ω to get $G \in V$ and s.t. $M_\omega[G] \models \mathrm{cf}(\lambda) = \omega$

$$V_{\lambda+1} \models \exists v \varphi(a, v), \text{ for all } a \in M_\omega[g_0] \cap V_{\lambda+1}.$$

Main Lemma

Assume ZFC $+ {I_0}^* + \mathrm{Det}_\lambda(\text{Z-Deg})$. Then $\forall u \subset \lambda$, $\forall P \subset \omega_1$,

$$\exists a, b \geq_\Gamma u \text{ s.t. } Z_b = Z_a \restriction P.$$

- $\mathrm{Con}(I_0) \Rightarrow \mathrm{Con}({I_0}^*)$.

  In fact, given a proper $j \in \mathcal{E}(L(V_{\lambda+1}))$, let $\mathbb{P}$ be Laver's poset for indestrucibility, and $G \subset \mathbb{P}$ be a V-generic filter, there is a proper $\bar{j} \in \mathcal{E}(L(V_{\lambda+1})^{V[G]})$ s.t. $\bar{j} \restriction L(V_{\lambda+1})^V = j$.

Proof of main lemma, sketch

Suppose NOT. Assume for some $P \subset \omega_1$, and some $u \subset \lambda$ s.t. $\neg\exists a, b \geq_\Gamma u$ s.t $Z_a = Z_b \restriction P$. Work in M_ω.

- Let δ be the least measurable cardinal of M_ω above λ.
- Let γ be the supremum of first δ many strongly inaccessible cardinals of M_ω above δ.
- Fix a $z \subseteq \gamma$ which codes a bijection $M_\omega | \gamma \to \gamma$.
- For $x \subset \gamma$, let Z^*_x be the set of first ω_1 Z-ordinals $\alpha > \gamma$.

Let $\mathbb{P}$ be the full product of the partial orders $\mathbb{P}_i$, $i < \delta$, where each $\mathbb{P}_i$ adds a generic subset to the i-th strongly inaccessible, β_i, of M_ω above δ.

- $\mathbb{P}$ preserves the $({<}\gamma)$-supercompactness of λ.
- This is witnessed by a tower of measures on $\mathscr{P}_\lambda(\eta)$, $\eta \in (\delta, \gamma)$. We are only interested in the ones in

$$I = \{\eta \in (\delta, \gamma) \mid \eta \text{ is strongly inaccessible}\}.$$

Let $\tau \in (M_\omega)^{\mathbb{P}}$ and $p_0 \in \mathbb{P}$ be such that

$$p_0 \Vdash \tau \text{ is a tower (indexed by } I) \text{ of measures as above}$$

In addition, p_0 decides the projected measure on $\mathscr{P}_\lambda(\delta)$.
Let $a_0 \subset \gamma$ be a set in M_ω which codes p_0 and $M_\omega|\gamma$.

Lemma (ZFC)

There is a $q \in \mathbb{P}$ such that $q \leq p_0$ and $q \Vdash Z^*_{(a,G)} = Z^*_a \restriction P$.

So one can choose two conditions $p, q \in \mathbb{P}$ below p_0 such that

1. $p \Vdash Z^*_a = Z^*_{a_0}$, where $a = (a_0, G)$.
2. $q \Vdash Z^*_b = Z^*_{a_0} \restriction P$, where $b = (a_0, G)$.

Using homogeneity of $\mathbb{P}$, choose M_ω-generics, $G_p, G_q \in V$, s.t.

1. $M_\omega[G_p] = M_\omega[G_q]$,
2. $p \in G_p$ and $q \in G_q$.

τ^{G_p} and τ^{G_q} project to the same measure on $\mathscr{P}_\lambda(\delta)$.

Next we use a mixed Prikry tower forcing, $\mathbb{Q} = \mathbb{Q}(\mu, \tau^G)$, where μ is the measure on δ. A $\mathbb{Q}$-generic gives a countable sequence $\langle(\eta_i, A_i) : i < \omega\rangle$, where

- $\langle \eta_i : i < \omega\rangle$ is a Prikry sequence for the normal measure μ,
- $\langle A_i : i < \omega\rangle$ is a diagonal Prikry sequence for $\langle \nu_i : i < \omega\rangle$, where ν_i is the fine normal measure on $\mathscr{P}_\lambda(\beta_{\eta_i})$ given by τ^G.

$\mathbb{Q}$ collapses γ to λ and makes $\mathrm{cf}(\lambda) = \omega$.

Choose $\mathbb{Q}$-generics H_p over $M_\omega[G_p]$ and H_q over $M_\omega[G_q]$ in the same manner with respect to τ.

- $\mathbb{Q}$ is λ-good. So H_p and H_q can be found in V.
- H_p, H_q project to the same δ-supercompact Prikry generic on $\mathscr{P}_\lambda(\delta)$. Call this generic H. In $M_\omega[H]$, $\mathrm{cf}(\lambda) = \omega$.

Let

1. a^* be the subset of λ given by $(M_\omega[G_p][H_p]|\gamma, a_0)$,
2. b^* be the subset of λ given by $(M_\omega[G_q][H_q]|\gamma, a_0)$.

Thus $Z_{a^*} = Z_{b^*} \restriction P$. The key point is that a^* and b^* compute every set in $V_{\lambda+1} \cap M_\omega[H]$.

By Generic Absoluteness, there is a pair (P, u) s.t.

$$\varphi(P, u) =_{\mathsf{def}} \neg(\exists a, b \geq_\Gamma u)\,(Z_a = Z_b \restriction P),$$

in $M_\omega[H]$. But we just produced a pair (a, b) s.t.

$$a, b \geq_\Gamma u \wedge Z_a = Z_b \restriction P.$$

Contradiction!

Q.E.D.

Thank you!