Topologies on ordinals and stationary reflection

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Outline

1. Topologies on ordinals
   - $\xi$-stationary sets

2. Indescribable cardinals

3. Polymodal Provability Logics
   - The Logic GLP
A measure of the structural richness of $OR$ is given by the complexity (largeness) of the (non-discrete) topologies, extending the usual order-topology, one can have on $OR$, or on some limit ordinal $\delta$.

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So let us consider the following increasing sequence of natural topologies on ordinals.
Recall that for $\delta$ a limit ordinal, the interval topology on $\delta$ is the topology generated by the set $B_0$ consisting of $\{0\}$ and the intervals $(\alpha, \beta)$.

Notice that $\tau_0$ is a Hausdorff scattered topology in which $0$ and all successor ordinals less than $\delta$ are isolated points.

We will define a sequence of topologies $\tau_0 \subseteq \tau_1 \subseteq \ldots \tau_\xi \subseteq \ldots$ on $\delta$, with $\tau_0$ being the interval topology.
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Topologies on ordinals

Given $\tau_\xi$, let $d_\xi : \mathcal{P}(\delta) \to \mathcal{P}(\delta)$ be the Cantor derivative operator, defined by:

$$d_\xi(A) = \{ \alpha < \delta : \alpha \text{ is an accumulation point of } A \text{ in the } \tau_\xi \text{ topology} \}.$$

Then let $\tau_{\xi+1}$ be the topology generated by

$$\mathcal{B}_{\xi+1} := \mathcal{B}_\xi \cup \{ d_\xi(A) : A \subseteq \delta \}.$$  

Notice that $d_0(A)$ is the set of limit points of $A$ in the ordinal ordering. Thus, if the cofinality of $\alpha$ is uncountable and $\alpha \in d_0(A)$, then $d_0(A) \cap \alpha$ is a club (closed and unbounded) subset of $\alpha$. 

Topologies on ordinals

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The set $B_1 := B_0 \cup \{d_0(A) : A \subseteq \delta\}$ is a base for a topology $\tau_1$ on $\delta$, known as the club topology.

Note that every $\alpha < \delta$ of countable cofinality is an isolated point of $\tau_1$.

It is easily seen that for every $A \subseteq \delta$,

$$d_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$
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$$d_1(A) = \{\alpha : A \cap \alpha \text{ is stationary in } \alpha\}.$$
As a warm-up for the general case, let us look at the conditions under which the topology $\tau_2$, generated by $\mathcal{B}_2 := \mathcal{B}_1 \cup \{ d_1(A) : A \subseteq \delta \}$, is non-discrete.

If $\alpha < \delta$ and some stationary subset $S$ of $\alpha$ does not reflect (i.e., $d_1(S) = \{\alpha\}$), then $\alpha$ is an isolated point of $\tau_2$. So, for $\tau_2$ to be non-discrete topology we need at least that some $\alpha < \delta$ is stationary-reflecting, i.e., $d_1(S) \cap \alpha \neq \emptyset$, for all stationary $S \subseteq \alpha$.

It is well-known that the first stationary-reflecting cardinal, if it exists, must be either weakly inaccessible or the successor of a singular cardinal.

So if, e.g., $\delta \leq \aleph_{\omega+1}$, then $\tau_2$ is discrete.
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It is well-known that the first stationary-reflecting cardinal, if it exists, must be either weakly inaccessible or the successor of a singular cardinal.

So if, e.g., $\delta \leq \aleph_{\omega + 1}$, then $\tau_2$ is discrete.
But for $\tau_2$ to be non-discrete we need more than just the existence of a stationary-reflecting cardinal $\alpha < \delta$. What we need is some $\alpha < \delta$ such that every pair $A, B$ of stationary subsets of $\alpha$ simultaneously reflect, that is, there exists $\beta < \alpha$ with $\beta \in d_1(A) \cap d_1(B)$. Let us call such an $\alpha$ simultaneously stationary-reflecting, or s-reflecting for short.
**Proposition**

$\mathcal{B}_2$ is a sub-base for a topology on $\delta$ such that for every $\alpha$, $\alpha$ is not isolated if and only if it is $s$-reflecting. Hence, $\tau_2$ is a non-discrete topology on $\delta$ if and only if some $\alpha < \delta$ is $s$-reflecting.
It is easy to see, using the characterization of weakly-compact cardinals in terms of $\Pi^1_1$ indescribability, that every weakly compact cardinal is s-reflecting. Thus, in every model of set theory where there exists a weakly compact cardinal less than some limit ordinal $\delta$, $\tau_2$ is a non-discrete topology on $\delta$.

R. Jensen$^1$ showed that in the constructible universe $L$ a cardinal $\kappa$ is stationary-reflecting if and only if it is weakly compact, hence if and only if it is s-reflecting.

Thus, in $L$, the set $\mathcal{B}_2$ is a base for the non-discrete $\tau_2$ topology on a limit ordinal $\delta$ if and only if there exists a weakly-compact cardinal less than $\delta$.

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Let us see now what are the general conditions under which the topologies $\tau_{\xi}$ are non-discrete. We begin with some definitions that generalize the notions of stationary set and stationary reflection.

**Definition**

Let $\delta$ be a limit ordinal. We say that $A \subseteq \delta$ is 0-stationary in $\alpha$ if and only if $A \cap \alpha$ is unbounded in $\alpha$.

For $\xi > 0$, we say that $A$ is $\xi$-stationary in $\alpha < \delta$ if and only if for every $\zeta < \xi$, every subset $S$ of $\alpha$ that is $\zeta$-stationary in $\alpha$ $\zeta$-reflects to some $\beta \in A$, i.e., $S \cap \beta$ is $\zeta$-stationary in $\beta$. 
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\( \xi \)-stationary sets

Note that \( A \) is 1-stationary in \( \alpha \) if and only if \( A \cap \alpha \) is stationary in \( \alpha \).

Clearly, if \( A \) is \( \xi \)-stationary in \( \alpha \), then \( A \) is also \( \zeta \)-stationary in \( \alpha \), for all \( \zeta < \xi \).

We have that for every \( \xi \),

\[
d_\xi(A) = \{ \alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha \}.
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Note that $A$ is 1-stationary in $\alpha$ if and only if $A \cap \alpha$ is stationary in $\alpha$.

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$$d_\xi(A) = \{ \alpha : A \cap \alpha \text{ is } \xi\text{-stationary in } \alpha \}.$$
\(\xi\)-stationary reflection

**Definition**

We say that a limit ordinal \(\alpha\) is \(\xi\)-stationary-reflecting \((\xi\text{-reflecting}, \text{for short})\) if and only if \(d\xi(S)\) is \(\zeta\)-stationary in \(\alpha\), for every \(\zeta < \xi\) and every \(S \subseteq \alpha\) that is \(\zeta\)-stationary in \(\alpha\).

It is easy to see that \(\alpha\) is 0-reflecting if and only if it is a limit ordinal; it is 1-reflecting if and only if it has uncountable cofinality; and it is 2-reflecting if and only if it is stationary-reflecting.
We say that a limit ordinal \( \alpha \) is \( \xi \)-stationary-reflecting (\( \xi \)-reflecting, for short) if and only if \( d_\xi(S) \) is \( \zeta \)-stationary in \( \alpha \), for every \( \zeta < \xi \) and every \( S \subseteq \alpha \) that is \( \zeta \)-stationary in \( \alpha \).

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We say that an ordinal $\alpha$ is $\xi$-simultaneously-stationary-reflecting (or $\xi$-s-reflecting, for short) if and only for every $\zeta < \xi$, every pair of $\zeta$-stationary subsets $A, B \subseteq \alpha$ simultaneously $\zeta$-reflect at some $\beta < \alpha$, i.e., $A \cap \beta$ and $B \cap \beta$ are $\zeta$-stationary in $\beta$.

Note that $\alpha$ is 1-s-reflecting if and only if it has uncountable cofinality; and it is 2-s-reflecting if and only if it is s-reflecting.

One can show that $\alpha$ is $\xi$-s-reflecting if and only if $d_\zeta(A) \cap d_\zeta(B)$ is $\zeta$-stationary in $\alpha$, for every $\zeta < \xi$ and every $\xi$-stationary $A, B \subseteq \alpha$. 
Definition

We say that an ordinal $\alpha$ is $\xi$-simultaneously-stationary-reflecting ($\xi$-s-reflecting, for short) if and only for every $\zeta < \xi$, every pair of $\zeta$-stationary subsets $A, B \subseteq \alpha$ simultaneously $\zeta$-reflect at some $\beta < \alpha$, i.e., $A \cap \beta$ and $B \cap \beta$ are $\zeta$-stationary in $\beta$.

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**Definition**

We say that an ordinal \(\alpha\) is \(\xi\)-simultaneously-stationary-reflecting (\(\xi\)-s-reflecting, for short) if and only for every \(\zeta < \xi\), every pair of \(\zeta\)-stationary subsets \(A, B \subseteq \alpha\) simultaneously \(\zeta\)-reflect at some \(\beta < \alpha\), i.e., \(A \cap \beta\) and \(B \cap \beta\) are \(\zeta\)-stationary in \(\beta\).

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One can show that \(\alpha\) is \(\xi\)-s-reflecting if and only if \(d_\zeta(A) \cap d_\zeta(B)\) is \(\zeta\)-stationary in \(\alpha\), for every \(\zeta < \xi\) and every \(\xi\)-stationary \(A, B \subseteq \alpha\).
Characterizing non-discreteness

We have the following characterization of the conditions under which $B_n$ is a base or a sub-base for a non-discrete topology.

**Theorem**

For every $\xi$, 

1. $B_\xi$ is a sub-base for a topology on $\delta$ such that for every $\alpha < \delta$, $\alpha$ is not isolated if and only if it is $\xi$-s-reflecting. Hence, $B_\xi$ generates a non-discrete topology on $\delta$ if and only if some $\alpha < \delta$ is $\xi$-s-reflecting.

2. $B_\xi$ is a base for a non-discrete topology on $\delta$ if and only if some $\alpha < \delta$ is $\xi$-reflecting and every $\xi$-reflecting $\alpha < \delta$ is $\xi$-s-reflecting.
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2. $B_{\xi}$ is a base for a non-discrete topology on $\delta$ if and only if some $\alpha < \delta$ is $\xi$-reflecting and every $\xi$-reflecting $\alpha < \delta$ is $\xi$-s-reflecting.
Recall that a cardinal \( \kappa \) is \( \Pi^1_n \)-indescribable if for every \( A \subseteq V_\kappa \) and every \( \Pi^1_n \)-sentence \( \varphi(A) \), if \( \langle V_\kappa, \in, A \rangle \models \varphi(A) \), then there is \( \lambda < \kappa \) such that \( \langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi(A \cap V_\lambda) \).

**Proposition**

Every \( \Pi^1_n \)-indescribable cardinal is \((n+1)\)-s-reflecting.

Thus, if there exists a \( \Pi^1_n \)-indescribable cardinal below some limit ordinal \( \delta \), then \( \mathcal{B}_{n+1} \) is a sub-base for a non-discrete topology on \( \delta \).
Recall that a cardinal $\kappa$ is $\Pi^1_n$-indescribable if for every $A \subseteq V_\kappa$ and every $\Pi^1_n$-sentence $\varphi(A)$, if $\langle V_\kappa, \in, A \rangle \models \varphi(A)$, then there is $\lambda < \kappa$ such that $\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi(A \cap V_\lambda)$.

**Proposition**

*Every $\Pi^1_n$-indescribable cardinal is $(n+1)$-s-reflecting.*

Thus, if there exists a $\Pi^1_n$-indescribable cardinal below some limit ordinal $\delta$, then $\mathcal{B}_{n+1}$ is a sub-base for a non-discrete topology on $\delta$. 
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Thus, if there exists a $\Pi^1_n$-indescribable cardinal below some limit ordinal $\delta$, then $\mathcal{B}_{n+1}$ is a sub-base for a non-discrete topology on $\delta$. 
Following Jensen $^2$, it is possible that one could show, as in the case of $\Pi_1^1$-indescribable cardinals and stationary-reflection, that in the constructible universe $L$, a cardinal is $(n + 1)$-reflecting if and only if it is $\Pi_n^1$-indescribable, and therefore if and only if it is $(n + 1)$-s-reflecting.

If this turns out to be the case, then in $L$ the $\Pi_n^1$-indescribable cardinals would be precisely the non-isolated points of the $\tau_{n+1}$ topology.
However, this is still open.

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Following Jensen \(^2\), it is possible that one could show, as in the case of \(\Pi^1_1\)-indescribable cardinals and stationary-reflection, that in the constructible universe \(L\), a cardinal is \((n + 1)\)-reflecting if and only if it is \(\Pi^1_n\)-indescribable, and therefore if and only if it is \((n + 1)\)-s-reflecting.

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Following Jensen \(^2\), it is possible that one could show, as in the case of \(\Pi_1^1\)-indecomposable cardinals and stationary-reflection, that in the constructible universe \(L\), a cardinal is \((n+1)\)-reflecting if and only if it is \(\Pi_n^1\)-indecomposable, and therefore if and only if it is \((n+1)\)-s-reflecting.

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\(\xi\)-indescribable cardinals

The so-called \(\xi\)-indescribable cardinals, introduced by Jensen, may be used for the general case.
For \(\xi > 0\), a cardinal \(\kappa\) is called \(\xi\)-indescribable if for every formula \(\varphi(x)\) of the first-order language of set theory, and any subset \(A \subseteq V_\kappa\), if

\[
\langle V_{\kappa+\xi}, \in, A \rangle \models \varphi(A)
\]

then for some \(\lambda < \kappa\),

\[
\langle V_{\lambda+\xi}, \in, A \cap V_\lambda \rangle \models \varphi(A \cap V_\lambda).
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Observe that \(\kappa\) is 1-indescribable if and only if it is \(\Pi^1_n\)-indescribable for every \(n\).

Jensen showed that if \(\kappa\) is the \(\omega\)-Erdös cardinal, then there are cardinals below \(\kappa\) that are \(\kappa\)-indescribable. Further, if \(\kappa\) is \(\xi\)-indescribable, then \(L \models "\kappa\) is \(\xi\)-indescribable"."
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Theorem

For $\xi > 0$, every $\xi$-indestructible cardinal $\kappa$ is $\xi$-s-reflecting.

So, if there exists a $\xi$-indestructible cardinal below some limit ordinal $\delta$, then the topology $\mathcal{T}_\xi$ on $\delta$ is non-discrete.
Theorem

For $\xi > 0$, every $\xi$-indescribable cardinal $\kappa$ is $\xi$-s-reflecting.

So, if there exists a $\xi$-indescribable cardinal below some limit ordinal $\delta$, then the topology $\tau_\xi$ on $\delta$ is non-discrete.
The ideal of non-$\xi$-stationary sets

For each limit ordinal $\alpha$ and each $\xi$, let $I_\alpha^\xi$ be the set of non-$\xi$-stationary subsets of $\alpha$, and let

$$F_\alpha^\xi = (I_\alpha^\xi)^* := \{ A \subseteq \alpha : \alpha \setminus A \in I_\alpha^\xi \}.$$

Thus, if $\alpha$ has uncountable cofinality, then $I_\alpha^1$ is the ideal of non-stationary subsets of $\alpha$ and $F_\alpha^1$ is the club filter over $\alpha$.

Proposition

For every $\xi$, an ordinal $\alpha$ is $\xi$-s-reflecting if and only if $I_\alpha^\xi$ is an ideal, hence if and only if $F_\alpha^\xi$ is a filter.
For each limit ordinal \( \alpha \) and each \( \xi \), let \( \mathcal{I}_\alpha^\xi \) be the set of non-\( \xi \)-stationary subsets of \( \alpha \), and let

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\mathcal{F}_\alpha^\xi = (\mathcal{I}_\alpha^\xi)^* := \{ A \subseteq \alpha : \alpha - A \in \mathcal{I}_\alpha^\xi \}.
\]

Thus, if \( \alpha \) has uncountable cofinality, then \( \mathcal{I}_\alpha^1 \) is the ideal of non-stationary subsets of \( \alpha \) and \( F_\alpha^1 \) is the club filter over \( \alpha \).

**Proposition**

For every \( \xi \), an ordinal \( \alpha \) is \( \xi \)-s-reflecting if and only if \( \mathcal{I}_\alpha^\xi \) is an ideal, hence if and only if \( \mathcal{F}_\alpha^\xi \) is a filter.
The ideal of non-$\xi$-stationary sets

For each limit ordinal $\alpha$ and each $\xi$, let $\mathcal{I}_\alpha^\xi$ be the set of non-$\xi$-stationary subsets of $\alpha$, and let

$$\mathcal{F}_\alpha^\xi = (\mathcal{I}_\alpha^\xi)^* := \{ A \subseteq \alpha : \alpha - A \in \mathcal{I}_\alpha^\xi \}.$$ 

Thus, if $\alpha$ has uncountable cofinality, then $\mathcal{I}_\alpha^1$ is the ideal of non-stationary subsets of $\alpha$ and $\mathcal{F}_\alpha^1$ is the club filter over $\alpha$.

**Proposition**

For every $\xi$, an ordinal $\alpha$ is $\xi$-s-reflecting if and only if $\mathcal{I}_\alpha^\xi$ is an ideal, hence if and only if $\mathcal{F}_\alpha^\xi$ is a filter.
The Logic \textbf{GLP}

Consider the language of propositional logic with additional modal operators $[n]$, for each $n \in \omega$. The corresponding dual operators $\neg[n]\neg$ are denoted by $\langle n \rangle$. The logic system \textbf{GLP} has the following axioms and rules:

\textbf{Axioms:}

1. Boolean tautologies.
2. $[n](\varphi \rightarrow \psi) \rightarrow ([n]\varphi \rightarrow [n]\psi)$, for all $n$.
3. $[n]([n]\varphi \rightarrow \varphi) \rightarrow [n]\varphi$, for all $n$.
4. $[m]\varphi \rightarrow [n]\varphi$, for all $m < n$.
5. $\langle m \rangle \varphi \rightarrow [n]\langle m \rangle \varphi$, for all $m < n$.

\textbf{Rules:}

1. $\vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$ (Modus Ponens)
2. $\vdash \varphi \Rightarrow \vdash [n]\varphi$, for all $n$ (Necessitation)
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1. $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$ (Modus Ponens)
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Let $\delta$ be a limit ordinal. A valuation on $\delta$ is a map \( \nu : \text{Form} \rightarrow \mathcal{P}(\delta) \) of formulas of GLP to subsets of $\delta$ such that:

1. \( \nu(\neg \varphi) = \delta - \nu(\varphi) \)
2. \( \nu(\varphi \land \psi) = \nu(\varphi) \cap \nu(\psi) \)
3. \( \nu(\langle n \rangle \varphi) = d_n(\nu(\varphi)), \) for all $n$. (Hence, \( \nu([n] \varphi) = \delta - d_n(\delta - \nu(\varphi)), \) for all $n$.)

Notice that

\[
\nu(\langle n \rangle \varphi) = \{ \alpha : \nu(\varphi) \cap \alpha \text{ has positive } F_n^\alpha \text{-measure} \}.
\]

\[
\nu([n] \varphi) = \{ \alpha : \nu(\varphi) \cap \alpha \in F_n^\alpha \}.
\]

A formula is valid in $\delta$ if \( \nu(\varphi) = \delta \), for every valuation $\nu$ on $\delta$. 
Let $\delta$ be a limit ordinal. A **valuation** on $\delta$ is a map $v : \text{Form} \to P(\delta)$ of formulas of **GLP** to subsets of $\delta$ such that:

1. $v(\neg \varphi) = \delta - v(\varphi)$
2. $v(\varphi \land \psi) = v(\varphi) \cap v(\psi)$
3. $v(\langle n \rangle \varphi) = d_n(v(\varphi))$, for all $n$. (Hence, $v([n] \varphi) = \delta - d_n(\delta - v(\varphi))$, for all $n$.)

Notice that

- $v(\langle n \rangle \varphi) = \{ \alpha : v(\varphi) \cap \alpha \text{ has positive } F_\alpha^n \text{-measure} \}$.
- $v([n] \varphi) = \{ \alpha : v(\varphi) \cap \alpha \in F_\alpha^n \}$.

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A formula is **valid** in $\delta$ if $\nu(\varphi) = \delta$, for every valuation $\nu$ on $\delta$. 
Soundness

Proposition

All the axioms of \textbf{GLP} are valid in $\delta$. 
Suppose there exist infinitely-many $\omega$-s-reflecting cardinals with $\sup \kappa$, and $\square_{\lambda}$ holds for all $\lambda < \kappa$. Then for every limit ordinal $\delta > \kappa$, every formula of the language of $\text{GLP}$ valid in $\delta$ is provable in $\text{GLP}$. 
The proof is based on the following Embedding Theorem, which generalizes similar theorems of Blass and Beklemishev for the case $n \leq 1$.
Let us write $R_i(x) := \{y : xR_iy\}$ and $\bar{R}_i(x) := R_i(x_i) \cup \bigcup_{i<j \leq n} R_j(x)$.

**Theorem**

Let $\kappa$ be as in the Theorem. If $\langle T, R_0, \ldots, R_n \rangle$ is a finite J-tree with root $r$, then there is an ordinal $\delta < \kappa$ and a map $S : T \to \mathcal{P}(\delta) \setminus \{\emptyset\}$ such that

1. $\{S_x : x \in T\}$ is pairwise disjoint, and for every $i \leq n$, $\{S_x : x \in T\}$ is pairwise disjoint, and for every $i \leq n$, $S_x \subseteq d_i(S_y)$. That is, if $\alpha \in S_x$, then $S_y \cap \alpha$ has positive $\mathcal{F}_\alpha^i$-measure.
2. $S_x \subseteq -d_i(- \bigcup_{y \in \bar{R}_i(x)} S_y)$, for all $i \leq n$. That is, if $\alpha \in S_x$, then $\bigcup_{y \in \bar{R}_i(x)} S_y \cap \alpha \in \mathcal{F}_\alpha^i$, for all $i \leq n$. 

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Topologies on Ordinals & Stationary Reflection