Some consistency results about saturated ideals

Sean Cox

Virginia Commonwealth University
scox9@vcu.edu

Ideals and generic ultrapower

- κ regular, uncountable
- $\mathcal{I} \subset \wp(\kappa)$ a **normal** ideal

EXAMPLE: $\mathcal{I} = \mathsf{NS}_\kappa$.

$\mathbb{B}_\mathcal{I} := \wp(\kappa)/\mathcal{I}$

Ideals and generic ultrapower

- κ regular, uncountable
- $\mathcal{I} \subset \wp(\kappa)$ a **normal** ideal

EXAMPLE: $\mathcal{I} = \mathrm{NS}_\kappa$.

$\mathbb{B}_\mathcal{I} := \wp(\kappa)/\mathcal{I}$

Fact

Forcing with $\mathbb{B}_\mathcal{I}$ yields a **generic ultrapower** $V \to_G ult(V, G)$.

Ideals and generic ultrapower

- κ regular, uncountable
- $\mathcal{I} \subset \wp(\kappa)$ a **normal** ideal

EXAMPLE: $\mathcal{I} = \mathsf{NS}_\kappa$.

$\mathbb{B}_\mathcal{I} := \wp(\kappa)/\mathcal{I}$

Fact

Forcing with $\mathbb{B}_\mathcal{I}$ yields a **generic ultrapower** $V \to_G ult(V, G)$.

"precipitous"

Saturation

Definition

$\mathcal{I}$ is **saturated** iff $\mathbb{B}_{\mathcal{I}}$ has κ^+-cc.

Theorem (Solovay?)

If $\mathcal{I}$ is saturated, then the corresponding generic ultrapower is almost huge.

Questions from *Handbook of Set Theory*

(paraphrasing)

Question (Foreman)

Is saturation equivalent to antichain catching?

Questions from *Handbook of Set Theory*

(paraphrasing)

Question (Foreman)

Is saturation equivalent to antichain catching?

Conjecture (Foreman)

If $\mathcal{I}$ is a saturated ideal on ω_2 then its dual concentrates on approachable ordinals.

Conjecture (Foreman)

If $\mathcal{I}$ is a saturated ideal on $\wp_{\omega_2}(H_\theta)$ then its dual concentrates on internally approachable sets.

Solutions

Theorem (C.-Zeman)

Antichain catching does **not** *imply saturation (or even strongness) of an ideal.*

Theorem (C.)

It is consistent relative to large cardinals that:

1. *There is a saturated ideal on ω_2 concentrating on non-approachable ordinals.*
2. *There is a saturated ideal on $\wp_{\omega_2}(H_\theta)$ concentrating on non-approachable sets.*

Outline

Antichain catching

1. Foreman-Magidor-Shelah
   - Martin's Maximum
   - Levy collapsing below supercompact
2. Woodin's stationary tower forcing

$S_{\mathcal{I}}^{\text{Catch}}$ denotes . . .

. . . the set of all $M \prec (H_{(2^\kappa)^+}, \in, \mathcal{I})$ such that:

- $\kappa_M := M \cap \kappa \in \kappa$
- Letting $\sigma : \bar{M} \to M \prec H_\theta$ be the inverse of Mostowski collapse, then the derived ultrafilter

$$\mathcal{U}_M := \{x \in P^{\bar{M}}(\kappa_M) \mid \kappa_M \in \sigma(x)\}$$

  is $(\bar{M}, \mathbb{B}_{\bar{\mathcal{I}}})$-**generic**.

How big is $S_{\mathcal{I}}^{\text{Catch}}$?

Definition (C.-Zeman)

Define:

$$\begin{aligned}
\text{ClubCatch}(\mathcal{I}) &:\iff S_{\mathcal{I}}^{\text{Catch}} \text{ is club} \\
\text{ProjectiveCatch}(\mathcal{I}) &:\iff S_{\mathcal{I}}^{\text{Catch}} \text{ is } \mathcal{I}\text{-projective stationary} \\
\text{StatCatch}(\mathcal{I}) &:\iff S_{\mathcal{I}}^{\text{Catch}} \text{ is stationary}
\end{aligned}$$

How big is $S_{\mathcal{I}}^{\mathsf{Catch}}$?

Definition (C.-Zeman)

Define:

$$\begin{aligned}
\mathsf{ClubCatch}(\mathcal{I}) \quad &:\Longleftrightarrow S_{\mathcal{I}}^{\mathsf{Catch}} \text{ is club} \\
\mathsf{ProjectiveCatch}(\mathcal{I}) \quad &:\Longleftrightarrow S_{\mathcal{I}}^{\mathsf{Catch}} \text{ is } \mathcal{I}\text{-projective stationary} \\
\mathsf{StatCatch}(\mathcal{I}) \quad &:\Longleftrightarrow S_{\mathcal{I}}^{\mathsf{Catch}} \text{ is stationary}
\end{aligned}$$

Theorem (Foreman-Magidor-Shelah)

$$\begin{array}{ccc}
\mathit{ClubCatch}(\mathcal{I}) & \Longleftrightarrow & \mathcal{I} \text{ saturated} \\
\Big\Downarrow A & & \Big\Downarrow \\
\mathit{ProjectiveCatch}(\mathcal{I}) & \overset{B}{\Longrightarrow} & \mathcal{I} \text{ is precipitous}
\end{array}$$

How big is $S^{\mathsf{Catch}}_{\mathcal{I}}$?

Definition (C.-Zeman)

Define:

$$\begin{aligned}
\mathsf{ClubCatch}(\mathcal{I}) &:\iff S^{\mathsf{Catch}}_{\mathcal{I}} \text{ is club}\\
\mathsf{ProjectiveCatch}(\mathcal{I}) &:\iff S^{\mathsf{Catch}}_{\mathcal{I}} \text{ is } \mathcal{I}\text{-projective stationary}\\
\mathsf{StatCatch}(\mathcal{I}) &:\iff S^{\mathsf{Catch}}_{\mathcal{I}} \text{ is stationary}
\end{aligned}$$

Theorem (Foreman-Magidor-Shelah)

$$\begin{array}{ccc}
\mathit{ClubCatch}(\mathcal{I}) & \Longleftrightarrow & \mathcal{I} \textit{ saturated}\\
\Big\Downarrow A & & \Big\Downarrow\\
\mathit{ProjectiveCatch}(\mathcal{I}) & \overset{B}{\Longrightarrow} & \mathcal{I} \textit{ is precipitous}
\end{array}$$

Question

Can either A or B be reversed?

ω_1 is special

Theorem (Ketchersid-Larson-Zapletal; Schindler)

If $\mathcal{I}$ is a normal ideal on ω_1, then TFAE:

1. *ProjectiveCatch*$(\mathcal{I})$
2. $\mathcal{I}$ *is precipitous*

Theorem (Ketchersid-Larson-Zapletal; Schindler)

If $\mathcal{I}$ is a normal ideal on ω_1, then TFAE:

1. *ProjectiveCatch*$(\mathcal{I})$
2. $\mathcal{I}$ *is precipitous*

Corollary

In the special case where $\mathcal{I} = NS_{\omega_1}$:

$$\begin{aligned} \mathcal{I} \text{ somewhere precipitous} &\iff S_{\mathcal{I}}^{Catch} \text{ is stationary} \\ \mathcal{I} \text{ precipitous} &\iff S_{\mathcal{I}}^{Catch} \text{ is projective stationary} \\ \mathcal{I} \text{ is saturated} &\iff S_{\mathcal{I}}^{Catch} \text{ is club} \end{aligned}$$

What about ω_2?

Theorem (C.-Zeman)

If there is an ideal $\mathcal{I}$ on ω_2 such that ProjectiveCatch($\mathcal{I}$) holds, then there is an inner model with a Woodin cardinal.

In particular: Precipitousness does not imply projective antichain catching for ideals on ω_2.

What about ω_2?

Theorem (C.-Zeman)

If there is an ideal $\mathcal{I}$ on ω_2 such that ProjectiveCatch($\mathcal{I}$) holds, then there is an inner model with a Woodin cardinal.

In particular: Precipitousness does not imply projective antichain catching for ideals on ω_2.

(recall these two notions are equivalent for ideals on ω_1)

Relation to Forcing Axioms

Lemma

Suppose $\mathcal{I}$ is ideal on ω_2. Then

$$\mathit{ProjectiveCatch}(\mathcal{I}) \implies \mathit{FA}_{\omega_1}(\mathbb{B}_{\mathcal{I}})$$

But the LHS has considerably more consistency strength.

“$\mathcal{J}$ canonically projects to $\mathcal{I}$”

EXAMPLE:

$$\mathcal{I} := \mathsf{NS}_{\omega_1} \subset \wp(\omega_1)$$

$$\mathcal{J} := \{A \subseteq \wp_{\omega_1}(H_\theta) \mid A \text{ is (generalized) nonstationary } \} \\ \subset \wp\Big(\wp_{\omega_1}(H_\theta)\Big)$$

"$\mathcal{J}$ canonically projects to $\mathcal{I}$"

EXAMPLE:

$$\mathcal{I} := \mathsf{NS}_{\omega_1} \subset \wp(\omega_1)$$

$$\mathcal{J} := \{A \subseteq \wp_{\omega_1}(H_\theta) \mid A \text{ is (generalized) nonstationary } \} \subset \wp\Big(\wp_{\omega_1}(H_\theta)\Big)$$

Fact

If $\mathcal{J}$ canonically projects to $\mathcal{I}$ then there is a natural Boolean homomorphism

$$h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$$

Equivalent definition of ProjectiveCatch($\mathcal{I}$)

Lemma

ProjectiveCatch($\mathcal{I}$) is equivalent to:

$$(\exists \mathcal{J})\big(\textit{supp}(\mathcal{J}) \textit{ is large and } h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}} \textit{ is } \textbf{regular}\big)$$

Saturation and ProjectiveCatch

Theorem (Foreman)

The following are equivalent:

- $\mathcal{I}$ *is saturated*
- $h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$ *is regular, where* $\mathcal{J}$ *is the "canonical club filter for* $\mathcal{I}$*"*

In other words: *iff* the "$\exists \mathcal{J}$" in the definition of ProjectiveCatch($\mathcal{I}$) is witnessed by a very particular ideal.

Saturation and ProjectiveCatch

Theorem (Foreman)

The following are equivalent:

- $\mathcal{I}$ *is saturated*
- $h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$ *is regular, where* $\mathcal{J}$ *is the "canonical club filter for* $\mathcal{I}$*"*

In other words: *iff* the "$\exists \mathcal{J}$" in the definition of ProjectiveCatch($\mathcal{I}$) is witnessed by a very particular ideal.

Question (Foreman, HST)

For the $\Leftarrow$ *direction: What if we don't assume* $\mathcal{J}$ *is the canonical club filter?*

Saturation and ProjectiveCatch

Theorem (Foreman)

The following are equivalent:

- *$\mathcal{I}$ is saturated*
- *$h_{\mathcal{I},\mathcal{J}} : \mathbb{B}_{\mathcal{I}} \to \mathbb{B}_{\mathcal{J}}$ is regular, where $\mathcal{J}$ is the "canonical club filter for $\mathcal{I}$"*

In other words: *iff* the "$\exists \mathcal{J}$" in the definition of ProjectiveCatch($\mathcal{I}$) is witnessed by a very particular ideal.

Question (Foreman, HST)

For the $\Leftarrow$ direction: What if we don't assume $\mathcal{J}$ is the canonical club filter?

Answer (C.-Zeman)

Yes. ProjectiveCatch($\mathcal{I}$) does not even imply that $\mathcal{I}$ is strong.

Antichain catching without saturation

Theorem (C.-Zeman)

Suppose κ is δ-supercompact, where δ is the **least** *inaccessible above κ. Then in $V^{Col(\omega_1,<\kappa)*Col(\kappa,<\delta)}$ there is an ideal $\mathcal{I}$ on ω_2 such that ProjectiveCatch($\mathcal{I}$) holds, yet $\mathcal{I}$ is not saturated (or even strong).*

REMARK 1: Best known upper bound for strong/saturated ideal on ω_2 is almost huge cardinal.

Antichain catching without saturation

Theorem (C.-Zeman)

Suppose κ is δ-supercompact, where δ is the **least** *inaccessible above κ. Then in $V^{Col(\omega_1,<\kappa)*Col(\kappa,<\delta)}$ there is an ideal $\mathcal{I}$ on ω_2 such that ProjectiveCatch($\mathcal{I}$) holds, yet $\mathcal{I}$ is not saturated (or even strong).*

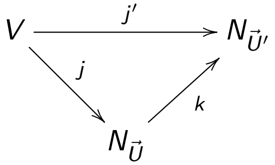
REMARK 1: Best known upper bound for strong/saturated ideal on ω_2 is almost huge cardinal.

REMARK 2: For ideals on ω_1, the solution to Foreman's question is MUCH simpler.

A non-saturated ideal on ω_2 satisfying ProjectiveCatch

Idea: mimic Kunen/Magidor forcing of a saturated ideal on ω_2 from an almost huge tower embedding.
Problem: To avoid creating a strong ideal, we were compelled to work with tower embedding which is NOT almost huge.

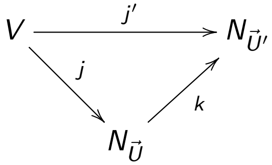
- This kills the ability to lift factorings of embeddings into generic extensions



A non-saturated ideal on ω_2 satisfying ProjectiveCatch

Idea: mimic Kunen/Magidor forcing of a saturated ideal on ω_2 from an almost huge tower embedding.
Problem: To avoid creating a strong ideal, we were compelled to work with tower embedding which is NOT almost huge.

- This kills the ability to lift factorings of embeddings into generic extensions



Salvage: if $\mathrm{crit}(k) > \mathrm{lh}(\vec{U})$ then we can interpolate the lifting of j with the lifting of j'.

Outline

Approachability

A set M is *internally approachable* iff there exists a sequence $\vec{N}$ such that

- $\vec{N}$ is a filtration of M; i.e.
  - Each N_ξ has size $< |M|$;
  - $\vec{N}$ is $\subseteq$-increasing and continuous;
  - $M = \bigcup_\xi N_\xi$
- Every proper initial segment of $\vec{N}$ is an element of M

Approachability

A set M is *internally approachable* iff there exists a sequence $\vec{N}$ such that

- $\vec{N}$ is a filtration of M; i.e.
  - Each N_ξ has size $< |M|$;
  - $\vec{N}$ is $\subseteq$-increasing and continuous;
  - $M = \bigcup_\xi N_\xi$
- Every proper initial segment of $\vec{N}$ is an element of M

Fact

Every **countable** $M \prec_1 V$ *is internally approachable.*

For uncountable M?

Shelah's Approachability ideal

$I[\kappa]$ is an ideal on κ with a technical definition related to internal approachability.

Fact

- *$I[\omega_1]$ is never proper ideal*
- *CH implies that $I[\omega_2]$ is not proper (i.e. dual is trivial).*

Can $I[\omega_2]$ coincide with nonstationary ideal?

NOTATION: $S^m_n := \{\alpha < \omega_m \mid \mathrm{cf}(\alpha) = \omega_n\}$

Question (Shelah)

Can $I[\omega_2] = NS \restriction S^2_1$?

Answer (Mitchell)

Yes. In fact, their equality is equiconsistent with some degree of Mahlo cardinal.

Constraints on saturation?

Theorem (Shelah)

If $\mathcal{I}$ is a saturated ideal on ω_2, then $S^2_0 \in \mathcal{I}$.

Conjecture (Foreman, HST)

If $\mathcal{I}$ is a saturated ideal on ω_2, then $I[\omega_2] \cap dual(\mathcal{I}) \neq \emptyset$.

Constraints on saturation?

Theorem (Shelah)

If $\mathcal{I}$ is a saturated ideal on ω_2, then $S^2_0 \in \mathcal{I}$.

Conjecture (Foreman, HST)

If $\mathcal{I}$ is a saturated ideal on ω_2, then $I[\omega_2] \cap dual(\mathcal{I}) \neq \emptyset$.

Theorem (C.)

Foreman's conjecture is false (relative to consistency of almost huge cardinals).

A sufficient condition for $I[\omega_2] \cap \text{dual}(\mathcal{I}) = \emptyset \ldots$

…is to arrange that whenever $j : V \to_G N$ is a generic ultrapower by $\mathbb{B}_{\mathcal{I}}$ (which has critical point ω_2), then

(V, N) has the ω_1-approximation property

Roughly this means: if $b \in N$ is a cofinal branch through a height ω_1 tree from V, then $b \in V$.

Aside

Approximation property comes up all the time.

Adding clubs and collapsing with finite conditions

- (70s/80s, Baumgartner): adding a club to ω_1 with finite conditions
- (80s, Todorcevic): forcing with models as side conditions
- (early 2000s, S. Friedman and Mitchell independently): adding club to ω_2 with finite conditions
  - (S. Friedman): model of PFA with an inner model which misses a real but computes ω_2 correctly
  - (Mitchell): $I[\omega_2]$ can be NS $\restriction S^2_1$
- (2010, Neeman): simplified version, and further force PFA with finite conditions

Strong properness (**Caution:** literature not consistent)

Definition (Mitchell)

(roughly) $\mathbb{P}$ is **strongly proper** with respect to a model $M \prec_1 V$ iff $\mathbb{P}$ can be factored through $M \cap \mathbb{P}$.

Theorem (Mitchell)

(special case): If $\mathbb{P}$ is strongly proper with respect to stationarily many countable models, then $(V, V^{\mathbb{P}})$ has the ω_1-approximation property.

Strong properness (**Caution:** literature not consistent)

Definition (Mitchell)

(roughly) $\mathbb{P}$ is **strongly proper** with respect to a model $M \prec_1 V$ iff $\mathbb{P}$ can be factored through $M \cap \mathbb{P}$.

Theorem (Mitchell)

(special case): If $\mathbb{P}$ is strongly proper with respect to stationarily many countable models, then $(V, V^{\mathbb{P}})$ has the ω_1-approximation property.

Friedman showed that strong properness (unlike properness) is especially well-behaved in cross products; he used this extensively in his model of PFA.

Saturated ideals on ω_2

Magidor argument (building on Kunen): let $j : V \to N$ be almost huge with $\kappa = \text{crit}(j)$ and $j(\kappa) = \delta$.

1. $\mathbb{P}$: κ-cc, "universal Kunen collapse" to turn κ into $\aleph_2$
2. $j(\mathbb{P})$ absorbs not only $\mathbb{P}$, but $\mathbb{P} * \text{Col}(\kappa, < \delta)$.
3. In $V^{j(\mathbb{P})}$ can construct a $V^{\mathbb{P} * \text{Col}(\kappa, <\delta)}$-normal ultrafilter $\mathcal{W}$ on κ
   - In $V^{\mathbb{P} * \text{Col}(\kappa, <\delta)}$ use a name for $\mathcal{W}$ to define a filter $\mathcal{F}$ on κ
   - Since $\frac{j(\mathbb{P})}{\mathbb{P} * \text{Col}(\kappa, <\delta)}$ has $\delta = \aleph_3^{V^{\mathbb{P} * \text{Col}(\kappa, <\delta)}}$-cc, then $\mathcal{F}$ is saturated.

Saturated ideals on ω_2

Magidor argument (building on Kunen): let $j : V \to N$ be almost huge with $\kappa = \text{crit}(j)$ and $j(\kappa) = \delta$.

1. $\mathbb{P}$: κ-cc, "universal Kunen collapse" to turn κ into $\aleph_2$
2. $j(\mathbb{P})$ absorbs not only $\mathbb{P}$, but $\mathbb{P} * \text{Col}(\kappa, < \delta)$.
3. In $V^{j(\mathbb{P})}$ can construct a $V^{\mathbb{P} * \text{Col}(\kappa, < \delta)}$-normal ultrafilter $\mathcal{W}$ on κ
    - In $V^{\mathbb{P} * \text{Col}(\kappa, < \delta)}$ use a name for $\mathcal{W}$ to define a filter $\mathcal{F}$ on κ
    - Since $\frac{j(\mathbb{P})}{\mathbb{P} * \text{Col}(\kappa, < \delta)}$ has $\delta = \aleph_3^{V^{\mathbb{P} * \text{Col}(\kappa, < \delta)}}$-cc, then $\mathcal{F}$ is saturated.

Problem: Closure of these posets:

- forces the filter $\mathcal{F}$ to concentrate on approachable ordinals.
- But is central to the construction of $\mathcal{W}$ in step 3

Saturated ideals on ω_2

Magidor argument (building on Kunen): let $j : V \to N$ be almost huge with $\kappa = \mathrm{crit}(j)$ and $j(\kappa) = \delta$.

1. $\mathbb{P}$: κ-cc, "universal Kunen collapse" to turn κ into $\aleph_2$
2. $j(\mathbb{P})$ absorbs not only $\mathbb{P}$, but $\mathbb{P} * \mathrm{Col}(\kappa, < \delta)$.
3. In $V^{j(\mathbb{P})}$ can construct a $V^{\mathbb{P}*\mathrm{Col}(\kappa,<\delta)}$-normal ultrafilter $\mathcal{W}$ on κ
   - In $V^{\mathbb{P}*\mathrm{Col}(\kappa,<\delta)}$ use a name for $\mathcal{W}$ to define a filter $\mathcal{F}$ on κ
   - Since $\frac{j(\mathbb{P})}{\mathbb{P}*\mathrm{Col}(\kappa,<\delta)}$ has $\delta = \aleph_3^{V^{\mathbb{P}*\mathrm{Col}(\kappa,<\delta)}}$-cc, then $\mathcal{F}$ is saturated.

Problem: Closure of these posets:

- forces the filter $\mathcal{F}$ to concentrate on approachable ordinals.
- But is central to the construction of $\mathcal{W}$ in step 3

My solution: Find s.p. variations; modify step 3.

Assume $\theta^{<\theta} = \theta$ for simplicity.
Foreman also conjectured that if $\mathcal{J}$ is a θ^+-saturated ideal on $\wp_{\omega_2}(H_\theta)$ then its dual must concentrate on internally approachable sets.

Saturated ideals and IA_{ω_1}

Assume $\theta^{<\theta} = \theta$ for simplicity.
Foreman also conjectured that if $\mathcal{J}$ is a θ^+-saturated ideal on $\wp_{\omega_2}(H_\theta)$ then its dual must concentrate on internally approachable sets.

The proof of the earlier theorem is easily modified (assuming j is huge) to:

Theorem (C.)

It is consistent to have a saturated ideal on $\wp_{\omega_2}(H_\theta)$ such that IA_{ω_1} has measure zero.

Some questions

Well-known:

Question

Can $NS \restriction S^2_1$ *be saturated?*

Related:

Question

Can projective antichain catching hold for $NS \restriction S^2_1$*?*

Some questions

Well-known:

Question

Can $NS \restriction S^2_1$ be saturated?

Related:

Question

Can projective antichain catching hold for $NS \restriction S^2_1$?

Question

How does ZFC constrain saturated ideals on ω_2 (other than Shelah's constraint)?