Tameness from Large Cardinal Axioms

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Goal

Apply large cardinals to Abstract Elementary Classes to prove, in particular, Shelah's Categoricity Conjecture.

What is an Abstract Elementary Class?

$(K, \prec_K)$ is an Abstract Elementary Class (AEC) iff

0. every element of K is a $L(K)$ structure;
1. $\prec_K$ is a partial order on K;
2. if $M \prec_K N$, then $M \subseteq N$;
3. $(K, \prec_K)$ respects $L(K)$ isomorphisms;
4. if $M_0 \prec_K M_2$, $M_1 \prec_K M_2$, and $M_0 \subseteq M_1$, then $M_0 \prec M_1$;
5. suppose $\langle M_i \in K : i < \alpha \rangle$ is a $\prec_K$-increasing continuous chain, then
   5.1 $\cup_{i<\alpha} M_i \in K$ and, for all $i < \alpha$, we have $M_i \prec_K \cup_{i<\alpha} M_i$; and
   5.2 if there is some $N \in K$ so that, for all $i < \alpha$, we have $M_i \prec_K N$, then we also have $\cup_{i<\alpha} M_i \prec_K N$.; and
6. *(Lowenheim-Skolem number)* $LS(K)$ is the minimal $\lambda \geq |L(K)| + \aleph_0$ so for any $M \in K$ and $A \subset |M|$, there is some $N \prec_K M$ such that $A \subset |N|$ and $\|N\| \leq |A| + \lambda$.

Why Abstract Elementary Classes?

- The AEC axioms capture the model theoretic structure that exists *without the compactness theorem*.
- This makes it ideal to deal with the following contexts:
  - theories in $L_{\kappa,\omega}(Q_\lambda)$, where Q_λ is a quantifier that says "there are at least λ-many"
  - homogeneous model theory
  - groups and graphs without the full elementary substructure .
  - and more!

Shelah's Presentation Theorem

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Theorem (Shelah's Presentation Theorem)

If K is an abstract elementary class with $LS(K) = \lambda$, then there is a $L_1 \supset L(K)$, an L_1 theory T_1 of size λ, and a set of T_1 types Γ of size $\leq 2^\lambda$ so that the models of K are exactly

$$PC(T_1, \Gamma, L) = \{M^* \upharpoonright L : M^* \models T_1 \textit{ and } M \textit{ omits each } p \in \Gamma\}$$

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The T_1 and Γ don't give us much insight into the class, although this has some nice applications (EM models, stuff I'll do later, etc.)

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Shelah's Categoricity Conjecture

One of the main test questions in AECs is to generalize Morley's Categoricity Theorem.

Conjecture (Shelah's Eventual Categoricity Conjecture for Successors, SCC)

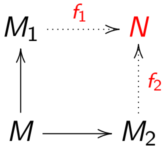
For every λ, there is some μ_λ so that if K is an AEC with $LS(K) = \lambda$ and is categorical in a successor cardinal greater than or equal to μ_λ, then it is categorical in every cardinal greater than or equal to μ_λ.

Morley's (and Shelah's) Categoricity Theorem says that, for first order theories, $\mu_\lambda = LS(K)^+ = |T|^+$.

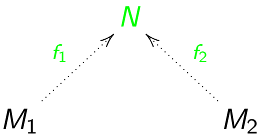
Simplifying assumptions

In order to simplify presentation, we assume that our AECs have the following properties:

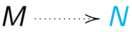
- **Amalgamation Property** - if $M \prec M_1, M_2$, then there is N and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$ so the following commutes:



- **Joint Embedding Property** - if $M_1, M_2 \in K$, then there is N and $f_\ell : M_\ell \to N$ for $\ell = 1, 2$



- **No Maximal Models** - every $M \in K$ has a proper extension N



These allow us to construct a monster model $\mathfrak{C}$, as in first order or homogeneous model theory.

Types lost

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Types as typically construed (sets of formulas) heavily use the compactness theorem, so they're worthless in AECs. Moreover, there's no natural syntax of our class to use.
However, one of the big consequences of types is the following semantic characterization:

$$tp(a/C) = tp(b/C) \iff \exists f \in Aut_C \mathfrak{C}. f(a) = b$$

This motivated Shelah's definition of *Galois types* .

Galois types found

The best conceptual way to think about Galois types

$$gtp(i/\mathbb{Q}, \mathbb{C}) = gtp(-i/\mathbb{Q}, \mathbb{C})$$

With our assumptions, we can say that a and b have the same type over M

$$gtp(a/M, \mathfrak{C}) = gtp(b/M, \mathfrak{C})$$

iff there is an $L(K)$-automorphism f of $\mathfrak{C}$ that fixes M pointwise and sends a to b.
We also use the notation $gS(M)$ to denote all Galois types over M.
Given $N \prec M$ and $p = tp(a/M) \in gS(M)$, $p \upharpoonright N \in gS(N)$ is $tp(a/N)$.

Tameness

In first order model theory, we know that any two types that are different differ on some finite subset of their domain,
If this holds in our AEC, we call it *tame*

Definition (Grossberg and VanDieren, 2006)

K is $< \kappa$ tame iff for every $M \in K$ and $p \neq q \in S(M)$, then there is $M_0 \prec M$ of size $< \kappa$ so $p \restriction M_0 \neq q \restriction M_0$.

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This property solves SCC for a lot of AECs.

Theorem (Grossberg and VanDieren)

Suppose K is an AEC that has amalgamation, joint embeddings, and no maximal models. If K is $< \kappa^+$-tame and λ^+ categorical for $\lambda \geq LS(K)^+ + \kappa$, then K is μ categorical for all $\mu \geq \lambda$.

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The rest of this talk argues that, with large cardinals, the answer is "lots" or even "all."

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Strongly Compact Cardinals

κ is **strongly compact** if each κ-complete filter can be extended to a κ-complete ultrafilter. More specifically, for each $\lambda \geq \kappa$, there is an ultrafilter on $P_\kappa \lambda := \{X \subset \lambda : |X| < \kappa\}$ that is fine; that is, for each $i \in \lambda$, we have $[i] := \{X \in P_\kappa \lambda : i \in X\}$ is in U.

We say that an AEC K is **essentially below** κ iff

- $LS(K) < \kappa$; or
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We say that an AEC K is **essentially below** κ iff

- $LS(K) < \kappa$; or
- K is axiomatized by some $L_{\kappa,\omega}$ theory.

Very relevant is the work of Makkai and Shelah that shows that, if κ is strongly compact, then $L_{\kappa,\omega}$ is tame (and a whole lot more). .

Łos' Theorem for AECs

The main technical result is a version of Łos' Theorem.

Theorem (Łos' Theorem for First Order Theories)

Let U be an ultrafilter over I, L be a language, and $\langle M_i : i \in I \rangle$ be L structures. Then, for any $[f_1]_U, \dots [f_n]_U \in \Pi M_i/U$ and $\phi(x_1, \dots, x_n) \in L$, we have

$$\Pi M_i/U \models \phi([f_1]_U, \dots, [f_n]_U) \textit{ iff } \{i \in I : M_i \models \phi(f_1(i), \dots, f_n(i))\} \in U$$

However, AECs lack any useful syntax or formulas, so our result in more cumbersome. .

Łos' Theorem for AECs

Theorem (Łos' Theorem for AECs, B.)

Let U be a κ complete ultrafilter on some index set I. If K is an AEC that is essentially below κ, then K and the class of K-embeddings is closed under κ complete ultrapowers and includes the ultrapower embedding.

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Let U be a κ complete ultrafilter on some index set I. If K is an AEC that is essentially below κ, then K and the class of K-embeddings is closed under κ complete ultrapowers and includes the ultrapower embedding. In particular,

1. *the ultraproduct of members of K is in K*
2. *taking the ultraproduct respects $\prec_K$*
3. *ultraproducts of isomorphic models are isomorphic*
4. *the average of embeddings is a map between ultraproducts*
5. *if U is nice enough[1], then M embeds naturally into its ultrapower*

[1] fine

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If $LS(K) < \kappa$, then all of the proofs essentially go the same way and rely on Shelah's Presentation Theorem.
For (1), if $M_i \in K$ for each $i \in I$, it has a good expansion M_i^*.
Then

$$\Pi M_i/U = \Pi(M_i^* \restriction L)/U = (\Pi M_i^*/U) \restriction L$$

By the theorem for $L_{\kappa,\omega}$, $\Pi M_i^*/U$ models T_1 and omits Γ, so $\Pi M_i/U \in K$.

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The rest of the parts follow similarly from extra clauses in the presentation theorem. .

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Theorem (B.)

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Let $a \vDash p$ and $b \vDash q$. For every "small" M_0, there is $f_{M_0} \in Aut_{M_0}\mathfrak{C}$ that sends a to b.

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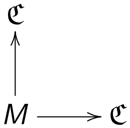
Use strong compactness to form a fine κ-complete ultrafilter U on $P^*_{\kappa+LS(K)^+}M := \{N \prec M : \|N\| < \kappa\}$. Fineness means $[m] := \{M_0 : m \in M_0\} \in U$ for each $m \in |M|$. .

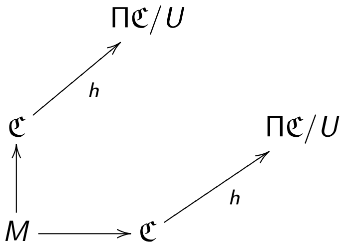
Tameness Everywhere

We have the following from set-up.



Tameness Everywhere

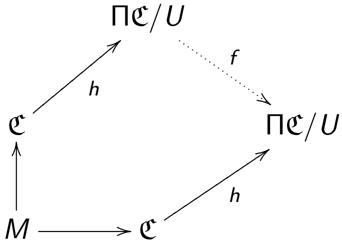
Each monster model can be embedded into its ultrapower by the ultrapower embedding h that sends x to $[M_0 \mapsto x]_U$



Tameness Everywhere

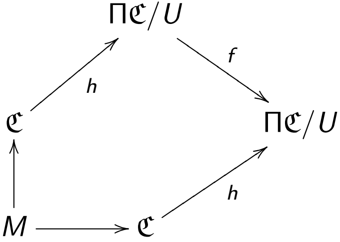
Our collection of automorphisms $\{f_{M_0} \in Aut_{M_0}\mathfrak{C} : M_0 \in P^*_{\kappa+LS(K)^+}M\}$ can be averaged into an automorphism of $\Pi\mathfrak{C}/U$ by taking

$$[M_0 \mapsto g(M_0)]_U \text{ to } [M_0 \mapsto f_{M_0}(g(M_0))]_U$$



Tameness Everywhere

$$f([M_0 \mapsto g(M_0)]_U) = [M_0 \mapsto f_{M_0}(g(M_0))]_U$$



This map

- fixes $h(M)$ by **fineness**
- send $h(a)$ to $h(b)$ since each f_{M_0} sends a to b

†.

Shelah's Categoricity Conjecture

Theorem (B.)

If there are class many strongly compacts, then SCC is true.

Note the hypothesis holds in V_θ if θ is extendible.
Proof: For each λ, set μ_λ, the threshold for categoricity, to be the successor of the first strongly compact above it.

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By Grossberg and VanDieren, it is categorical everywhere above χ.

By Shelah (not using any tameness assumptions), it is categorical everywhere between $\beth_{(2^{\beth_{(2^{LS(K)})^+}})^+}$ and χ, an interval that includes μ_λ. †.

More large cardinals

Theorem (B.)

- *If K is essentially below measurable κ, then K is $(< \lambda, \lambda)$ tame for cf $\lambda = \kappa$ (in particular, it's (κ, ∞) local)*
- *If $LS(K) < \kappa$, κ weakly compact, then K is $(< \kappa, \kappa)$ tame.*

These can be proved using elementary embedding and indescribability characterizations of large cardinals.
Also, Kolman and Shelah have some results on $L_{\kappa,\omega}$ for κ measurable that can be generalized to AECs essentially below measurable cardinals. .

Tameness Everywhere Always?

Can tameness be everywhere?

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- All AECs with some sort of structure theorem are tame to some degree.
- However, examples of AECs that are not tame have been constructed.
    - The Hart and Shelah examples, reexamined with tameness in mind by Baldwin and Kolesnikov, give examples of the failure of tameness below $\aleph_\omega$ in ZFC.
    - Baldwin and Shelah created an example of non-$(< \kappa, \kappa)$ from an almost free, non-free, non-Whitehead group of size κ. The existence of this is dependent on set theory for $\kappa > \aleph_1$.

So the existence of a threshold cardinal for tameness at least implies $V \neq L$.

Thanks!

Thanks!