Initial structures in the Tukey types of non-p-points

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References

[Blass/Dobrinen/Raghavan] *The next best thing to a p-point*, preprint.

[Dobrinen] *Initial structures in the Tukey types of non-p-points*, in preparation.

[Dobrinen/Todorcevic 1] *Tukey types of ultrafilters*, Illinois Journal of Mathematics, to appear.

[Dobrinen/Todorcevic 2] *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 1*, Transactions of the American Mathematical Society, to appear.

[Dobrinen/Todorcevic 3] *A new class of Ramsey-classification Theorems and their applications in the Tukey theory of ultrafilters, Part 2*, Transactions of the American Mathematical Society, to appear.

[Raghavan] *The generic ultrafilter added by* $(\mathrm{Fin}\times\mathrm{Fin}^+)$ ArXiv:1210.7387.

[Raghavan/Todorcevic] *Cofinal types of ultrafilters*, Annals of Pure and Applied Logic, 2012.

The Forcing $\mathbb{P} = (\mathsf{Fr} \otimes \mathsf{Fr})^+$

$X \in \mathbb{P}$ iff $X \subseteq \omega \times \omega$ and infinitely many fibers of X are infinite.

$Y \leq X$ iff $Y \setminus X \in \mathsf{Fin} \otimes \mathsf{Fin}$.

$\mathcal{P}(\omega \times \omega)/(\mathsf{Fin} \otimes \mathsf{Fin})$ is the separative quotient of $(\mathbb{P}, \leq)$.

$\mathbb{P}$ forces a generic ultrafilter $\mathcal{G}$ on $\omega \times \omega$.

Thm. [Blass in [BDR]]

1. $\pi_1 : \omega \times \omega \to \omega$ is not finite-to-one or constant on any set in $\mathcal{G}$. Thus, $\mathcal{G}$ is not a p-point.
2. $\mathcal{G}$ is a q-point and a weak p-point.
3. $\pi_1(\mathcal{G})$ is a selective ultrafilter, and is generic for $\mathcal{P}(\omega)/\text{Fin}$.

Def. $\mathcal{U}$ is a *p-point* if for each $f : \text{base}(\mathcal{U}) \to \omega$, there is a $U \in \mathcal{U}$ such that $f \restriction U$ is constant or finite-to-one.

q-point if for each finite-to-one $f : \text{base}(\mathcal{U}) \to \omega$, there is a $U \in \mathcal{U}$ such that $f \restriction U$ is 1-1.

selective if for each $f : \text{base}(\mathcal{U}) \to \omega$, there is a $U \in \mathcal{U}$ such that $f \restriction U$ is constant or 1-1.

Thm. [Blass in [BDR]] If $f : \omega \times \omega \to \omega$, then there is an $X \in \mathcal{G}$ such that $f \upharpoonright X$ is one of the following:

1. a constant function
2. π_1 followed by a 1-1 function
3. 1-1.

Cor. If $\mathcal{V} \leq_{RK} \mathcal{G}$, then $\mathcal{V}$ is either principal, isomorphic to $\pi_1(\mathcal{G})$, or isomorphic to $\mathcal{G}$.

Thm. [Blass in [BDR]] $\omega \times \omega \to [\mathcal{G}]^2_{4,3}$.

This is the best partition property a non-p-point can have.

Tukey Reducibility for Ultrafilters

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters.

Def. $\mathcal{X} \subseteq \mathcal{U}$ is *cofinal* in $(\mathcal{U}, \supseteq)$ iff $\mathcal{X}$ is a filter base for $\mathcal{U}$.

Def. $\mathcal{V}$ is *Tukey reducible* to $\mathcal{U}$ $(\mathcal{V} \le_T \mathcal{U})$

$\Leftrightarrow$ There is a *cofinal (convergent) map* from $\mathcal{U}$ into $\mathcal{V}$: $\exists\, f : \mathcal{U} \to \mathcal{V}$ mapping cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{D}$;

Fact. $\mathcal{U} \ge_T \mathcal{V} \Leftrightarrow$ there is a *monotone* cofinal map witnessing this; i.e. $X \supseteq Y \Rightarrow f(X) \supseteq f(Y)$.

Motivations for studying Tukey Reduction on the Class of Ultrafilters

1. A special class of directed systems of size $\mathfrak{c}$.
(In contrast to non-classification theorem of Todorcevic for directed posets of size $\mathfrak{c}$)

2. Tukey equivalence is a coarsening of the well-studied Rudin-Keisler equivalence: $\mathcal{U} \geq_{RK} \mathcal{V}$ implies $\mathcal{U} \geq_T \mathcal{V}$.

$\mathcal{U} \geq_{RK} \mathcal{V}$ iff there is a function $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$.

3. Isbell's Problem: Is there is more than 1 Tukey type?

Thm. [Isbell 65, Juhász 67] There is an ultrafilter $\mathcal{U}_{\text{top}}$ which has the maximum Tukey type: $(\mathcal{U}_{\text{top}}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.

Note. The Tukey type of $\mathcal{U}_{\text{top}}$ has cardinality $2^{\mathfrak{c}}$.

Question [Isbell 65]. Is there an ultrafilter with Tukey type strictly below the top?

Thm. [D/T 1] Every p-point is not Tukey top.

Def. An ultrafilter $\mathcal{U}$ is *basically generated* if it has a filter basis $\mathcal{B} \subseteq \mathcal{U}$ with the property that each sequence $\{A_n : n < \omega\} \subseteq \mathcal{B}$ converging to an element of $\mathcal{B}$ has a subsequence $\{A_{n_k} : k < \omega\}$ such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$.

Thm. [D/T 1] If $\mathcal{U}$ is basically generated, then $\mathcal{U}$ is not Tukey top.

Thm. [Raghavan in [R/T]] If $\mathcal{U}$ is basically generated, then there are only $\mathfrak{c}$ many $\mathcal{V} \leq_T \mathcal{U}$.

Def. The *Fubini product* of $\mathcal{U}$ and $\mathcal{V}_n$, $n < \omega$, is

$$\lim_{n \to \mathcal{U}} \mathcal{V}_n = \{X \subseteq \omega \times \omega : \mathcal{U}n \; X(n) \in \mathcal{V}_n\}$$

If all $\mathcal{V}_n = \mathcal{V}$, we write $\mathcal{U} \cdot \mathcal{V}$.

Thm. [D/T 1] The collection of all ultrafilters basically generated by some filter base closed under finite intersections includes all p-points and is closed under Fubini products.

Open Question: Is there a basically generated ultrafilter which is not a Fubini iterate of p-points?

Question [Blass]. Let $\mathcal{G}$ be the generic ultrafilter for $\mathcal{P}(\omega \times \omega)/(\mathrm{Fin} \otimes \mathrm{Fin})$.

1. Is $\mathcal{G}$ Tukey top?
2. Is $\mathcal{G}$ basically generated?

Let $\mathbb{P}_s$ denote the set of *standard* conditions in $\mathbb{P}$: those $X \subseteq \omega \times \omega$ all of whose non-empty fibers are infinite.

Let $\mathcal{G}_s$ denote the collection of the members of $\mathcal{G} \cap \mathbb{P}_s$. Note: $\mathcal{G}_s$ is a filter base for $\mathcal{G}$.

Def. Let $X \in \mathbb{P}_s$. A monotone map $f : \mathcal{G}_s \restriction X \to \mathcal{P}(\omega)$ is *represented by a monotone finitary map* φ if there is a map $\varphi : [\omega^2]^{<\omega} \to [\omega]^{<\omega}$ such that for all $s, t \in [\omega^2]^{<\omega}$,

1. (Monotonicity) $s \subseteq t \to \varphi(s) \subseteq \varphi(t)$;
2. (φ represents f) For each $U \in \mathcal{G}_s \restriction X$,

$$f(U) = \bigcup_{n<\omega} \varphi(U \cap (n \times n)).$$

Thm. [Canonization of monotone maps as represented by finitary maps] [Dobrinen in [BDR]]

In $V[\mathcal{G}]$, for each monotone function $f : \mathcal{G} \to \mathcal{P}(\omega)$, there is an $X \in \mathcal{G}_s$ such that $f \restriction (\mathcal{G}_s \restriction X)$ is represented by a monotone finitary map $\varphi : [X]^{<\omega} \to [\omega]^{<\omega}$.

Answering Question 1 of Blass:

Cor. [Dobrinen in [BDR]] $\mathcal{G}$ does not have maximal Tukey type.

Thm. [Raghavan in [BDR]] In $V[\mathcal{G}]$, let $\phi : \mathcal{G} \to \mathcal{P}(\omega)$ be a monotone non-zero map. Then there exist $P \subseteq \left[\omega^2\right]^{<\omega}$ and $\psi : P \to \omega$ such that

1. $\forall X \in \mathcal{G} \left[P \cap [X]^{<\omega} \neq \emptyset\right]$.
2. $\forall X \in \mathcal{G} \exists Y \in \mathcal{G} \cap [X]^{\omega} \forall s \in P \cap [Y]^{<\omega} [\psi(s) \in \phi(Y)]$.

Thm. [Raghavan in [BDR]] In $V[\mathcal{G}]$, let $\mathcal{V} \leq_T \mathcal{G}$. Then there is $P \subseteq \left[\omega^2\right]^{<\omega} \setminus \{\emptyset\}$ such that

1. $\forall t, s \in P \, [t \subseteq s \to t = s]$
2. $\mathcal{G}(P) \equiv_T \mathcal{G}$
3. $\mathcal{V} \leq_{RK} \mathcal{G}(P)$.

Answering Question 1 of Blass:

Cor. [Raghavan in [BDR]] $\mathcal{G}$ does not have maximal Tukey type.

Answering Question 2 of Blass:

Thm. [Raghavan in [BDR]] $\mathcal{G}$ is not basically generated.

Answering a question of Raghavan in [R]:

Thm. [Dobrinen in [BDR]] $(\mathcal{G}, \supseteq) \not\geq_T ([\omega_1]^{<\omega}, \subseteq)$.

Now that we know $\mathcal{G}$ is not at the top of the Tukey hierarchy, where actually is it?

Question. What is Tukey reducible to $\mathcal{U}$?

Know: If $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{V}$ is either isomorphic to one of $\mathcal{U}$, $\pi_1(\mathcal{U})$, or else is principal.

Is something similar for Tukey?

The rest of the work in this talk is due to Dobrinen and will appear in [D].

Thm. [D] If $\mathcal{V} \leq_T \mathcal{G}$, then $\mathcal{V}$ is isomorphic to a tree ultrafilter, all of whose nodes branch into an ultrafilter Tukey equivalent to one of $\mathcal{G}$, $\pi_1(\mathcal{G})$, or 1.

MainThm. [D] If $\mathcal{V} \leq_T \mathcal{G}$, then

1. $\mathcal{V} \equiv_T \mathcal{G}$; or
2. $\mathcal{V} \equiv_T \pi_1(\mathcal{G})$; or
3. $\mathcal{V}$ is principal.

This uses the following theorem (proved for rapid p-points in [D/T 1]).

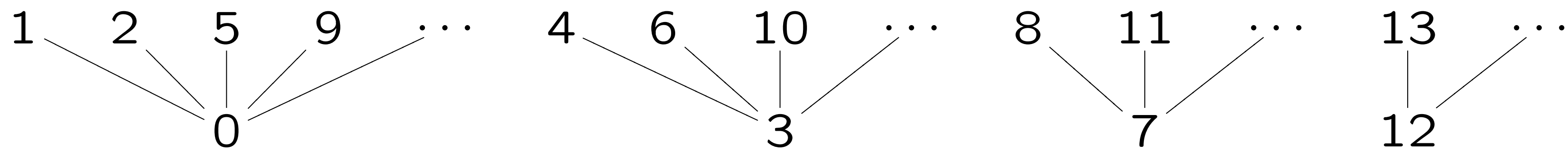
Thm. [D] $\mathcal{G} \cdot \mathcal{G} \equiv_T \mathcal{G}$.

Proof Outline

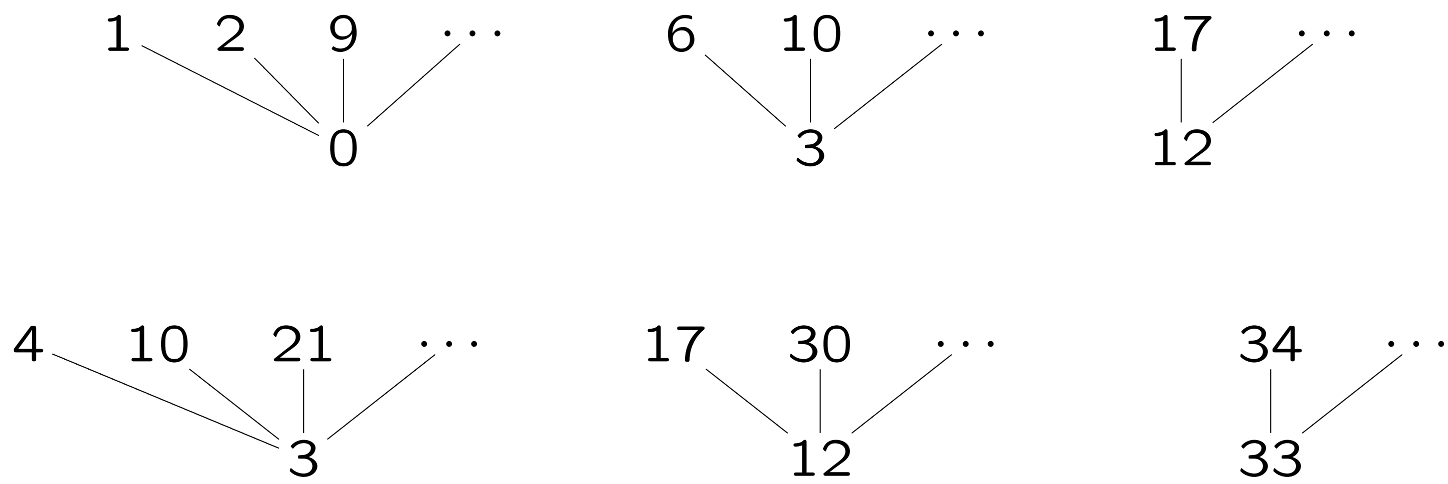
1. Pair down the forcing $\mathbb{P}$ to a dense subspace $(\mathcal{R}, \leq, r)$ which satisfies all axioms for a topological Ramsey space, except A.3(b).
2. Prove a diagonalization lemma.
3. Prove the Pigeonhole Property for 1-extensions of finite approximations.
4. Prove the Nash-Williams theorem for fronts on $\mathcal{R}$.
5. Use it to find canonical equivalence relations.
6. Apply these along with Canonical Finitary Maps Thm to analyze the which ultrafilters are Tukey reducible to $\mathcal{G}$.

Uses some ideas from [D/T 2], but has to be reworked.

The maximal member $\mathbb{W} \cong \omega^2$ of $\mathcal{R}$



Examples of members of $\mathcal{R}$



$X \in \mathcal{R}$ iff

1. $X \subseteq \mathbb{W}$;
2. If $i \in \pi_1[X]$, then $\{j \in \omega : (i,j) \in X\}$ is infinite;
3. $X(0) < X(0,0) < X(0,1) < X(1) < X(1,1) < X(0,2) < X(1,2) < X(2) < X(2,2) < X(0,3) < X(1,3) < X(2,3) < X(3) < X(3,3) < \ldots$.
   (i.e. $X \cong \mathbb{W}$)

For $X, Y \in \mathcal{R}$, define $Y \leq X$ iff $Y \subseteq X$.

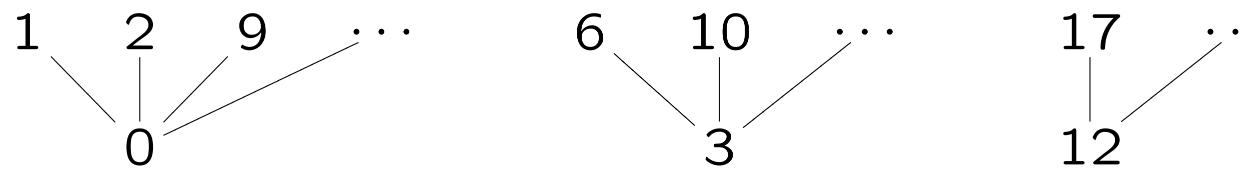
Note: $\mathcal{R}$ is dense in $\mathbb{P}$.

For $X \in \mathcal{R}$, define finite approximations $r_n(X)$ as follows.

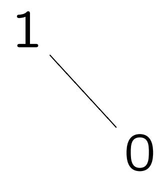
$$
\begin{aligned}
r_0(X) &= \emptyset \\
r_1(X) &= \{X(0,0)\} \\
r_2(X) &= \{X(0,0), X(0,1), X(1,1)\} \\
r_3(X) &= \{X(0,0), X(0,1), X(0,2), X(1,1), X(1,2), X(2,2)\}
\end{aligned}
$$

$$
r_{n+1}(X) = r_n(X) \cup \{X(0,n), \ldots, X(n,n)\} \tag{1}
$$

$X =$

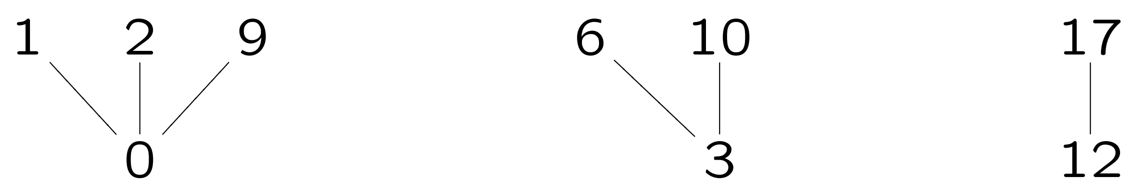


$r_1(X) =$



$r_2(X) =$

$r_3(X) =$



$\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$.

$\mathcal{AR} = \bigcup_{n<\omega} \mathcal{AR}_n$.

For $t \in \mathcal{AR}$, $[t, X] = \{Y \leq X : t \sqsubset Y\}$.

For $t \in \mathcal{AR}_n$, $r_{n+1}[t, X] = \{s \in \mathcal{AR}_{n+1} : t \sqsubset s \sqsubset X\}$.

Def. Given $t \in \mathcal{AR}_n$, a property $P(s,Y)$ for $s \in r_{n+1}[t, \mathbb{W}]$ is called *hereditary* if whenever $P(s,Y)$ holds, then also $P(s,Z)$ holds for all $Z \in [s,Y]$.

Lem. (Diagonalizing Hereditary Properties) [D] Suppose $P(s,Y)$ is a hereditary property. If for each $s \in \mathcal{AR}|X$ and each $Y \in [s,X]$ there is a $Z \in [s,Y]$ such that $P(s,Z)$, then there is an $A \leq X$ such that for each $s \in \mathcal{AR}|A$, $P(s,A)$.

Lem. (Pigeonhole Principle) [D] Let $n \geq 0$, $X \in \mathcal{R}$, $u \in \mathcal{AR}_n|X$, and $\mathcal{H} \subseteq r_{n+1}[u, X]$. Then there is a $Y \in [u, X]$ such that

1. $r_{n+1}[u, Y] \subseteq \mathcal{H}$; or
2. $r_{n+1}[u, Y] \cap \mathcal{H} = \emptyset$.

Def. A subset $\mathcal{F} \subseteq \mathcal{AR}$ is called a *front* if

1. For each $X \in \mathcal{R}$, there is an $a \in \mathcal{F}$ such that $a \sqsubseteq X$; and
2. For all $a, b \in \mathcal{F}$, $a \neq b \to a \not\sqsubseteq b$.

Thm. (Abstract Nash-Williams) [D] Let $t \in \mathcal{AR}$ and $X \in \mathcal{R}$ such that $[t, X] \neq \emptyset$. Let $\mathcal{F}$ be a front on $[t, X]$. Then for each $\mathcal{H} \subseteq \mathcal{F}$, there is a $Y \in [t, X]$ such that either $\mathcal{F}|Y \subseteq \mathcal{H}$ or else $\mathcal{F}|Y \cap \mathcal{H} = \emptyset$.

Thm. [D] There are canonizations of the equivalence relations on $r_{n+1}[n, X]$.

Finding the Tukey types of all $\mathcal{V} \leq_T \mathcal{U}$

Recall:

Thm. (Canonization of monotone maps as represented by finitary maps) [D in [BDR]]

In $V[\mathcal{G}]$, for each monotone function $g : \mathcal{G} \to \mathcal{P}(\omega)$, there is an $X \in \mathcal{G}_s$ such that $g \restriction (\mathcal{G}_s \restriction X)$ is represented by a monotone finitary map $f : [X]^{<\omega} \to [\omega]^{<\omega}$.

[D/T 2] There are a front $\mathcal{F}$ and an $f : \mathcal{F} \to \omega$ with $f(\mathcal{U}|\mathcal{F}) = \mathcal{V}$.

f induces equivalence relations on $r_{n+1}[t, X]$. They are canonical on some $A \in \mathcal{G}_s$.

Thm. [D] $f(\mathcal{U}|\mathcal{F})$ is isomorphic to a tree ultrafilter, all of whose nodes branch into some ultrafilter which is Tukey equivalent to one of $\mathcal{U}$, $\pi_1(\mathcal{U})$, or 1.

Similar theorems hold for the forcings $\mathcal{P}(\omega^n)/\mathsf{Fin}^{\otimes n}$ (modulo triple-checking).