Omitting types in infinitary $[0,1]$-valued logic

Christopher Eagle
University of Toronto

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Model theory for metric structures

Metric structures: $\langle M, d, (f_i)_{i\in I}, (R_j)_{j\in J}, (c_k)_{k\in K}\rangle$

- (M, d) metric space of diameter ≤ 1
  - Not necessarily complete
- $f_i : M^n \to M$, uniformly continuous
- $R_j : M^n \to [0, 1]$, uniformly continuous
- $c_k \in M$.

Example

First-order structures, (unit balls of) Banach spaces, operator algebras, . . .

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- Signatures specify moduli of uniform continuity for functions and predicates.
- Terms defined exactly as in first-order.
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  - $d(t_1(\overline{x}), t_2(\overline{x}))$
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  - Constant formula r for each $r \in \mathbb{Q} \cap [0,1]$.
- Formulas:
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- Formula $\phi(x_1, \ldots, x_n)$ defines a function on each structure M $\phi^M : M^n \to [0, 1]$.
- Write $M \models \phi(a_1, \ldots, a_n)$ to mean $\phi^M(a_1, \ldots, a_n) = 0$.
- $M \models \phi(\bar{a}) \to \psi(\bar{b}) \iff \phi^M(\bar{a}) \leq \psi^M(\bar{b})$.
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Infinitary $[0, 1]$-valued logic

- Goal: Extend to an infinitary logic with properties similar to $L_{\omega_1,\omega}$.
  - Scott sentences for separable structures.
  - Omitting types for countable fragments.
- Given formulas $\phi_0, \phi_1, \ldots$, allow a formula $\inf_{n<\omega} \phi_n$ (approximate infinitary disjunction).
  - Call this logic $L_{\omega_1,\omega}[0, 1]$.
  - If $\phi(x_1, \ldots, x_m)$ is an $L_{\omega_1,\omega}[0, 1]$-formula, then $\phi^M : M^n \to [0, 1]$ is not necessarily continuous!
  - Discontinuous formulas can be used to study structures from functional analysis, but care is needed.

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Examples of notions from Banach space theory expressible in $L_{\omega_1,\omega}[0,1]$ (in signatures expanding that of Banach spaces) include:

- A sequence (named as constant symbols) is a Schauder basis.
- Failure of hereditary indecomposibility.
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- Equivalence of norms.
- $\ldots$

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Scott sentences

Theorem

For each separable Banach space X there is an $L_{\omega_1,\omega}[0,1]$-sentence σ such that if Y is separable and $Y \models \sigma$, then Y is almost isometric to X.

- The proof is an analysis of the back-and-forth games introduced by Heinrich and Henson, analogous to Ehrenfeucht-Fraïssé games.
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Omitting Types - Definitions

- A **fragment** of $L_{\omega_1,\omega}[0,1]$ is a set of $L_{\omega_1,\omega}[0,1]$ formulas containing the atomic formulas and closed under substitutions, subformulas, finitary connectives, and (approximate) existential quantification.
- A **type** of a theory T is a set of formulas which is satisfied in some model of T.
- A type $\Sigma(\overline{x})$ is **principal** over T if there is a formula $\phi(\overline{x})$ and an $r \in \mathbb{Q} \cap (0,1)$ such that:
  - ϕ is consistent with T,
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- With a slightly modified definition of principal, this implies a version where the resulting structure is separable and complete.
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- Goal: Use topological ideas to give a simple proof.
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- C - countable set of new constant symbols,
- $L(C)$ - least countable fragment containing L and C,
- $\mathcal{W}$ - subspace of $Str_{L(C)}$ such that $M \models \inf_x \phi$ is witnessed in C.

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A sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of a space X is **complete** if the following holds:

- If: $\mathcal{F}$ is a centred family of closed sets
- And: for each n there is $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ such that $F_n \subseteq U_n$
- Then: $\bigcap \mathcal{F} \neq \emptyset$.

A completely regular space X is **Čech-complete** if it has a complete sequence of open covers.

Lemma

If X is Čech-complete then every closed subspace of X is Baire.

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$\mathcal{W}$ is Čech-complete.

Idea of proof.

- Enumerate the sentences as $\sigma_0, \sigma_1, \ldots$.
- The sets in $\mathcal{U}_n$ prescribe estimates for $\sigma_0, \ldots, \sigma_n$.
- Estimates at stage $n+1$ refine those from stage n.
- When estimating $\inf_{k<\omega} \phi_k$, also choose a k and make the same estimate for ϕ_k.
- Ultrafilter limits of sequences following these open covers exist. A Łoś-type result shows that for these sequences, ultrafilter limits can be computed as ultraproducts.
  - Similar to the proof of compactness in first-order, but this space is not compact!

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Application - Separable Quotients

Theorem

Let $T : X \to Y$ be a bounded surjective linear map, where X and Y are Banach spaces and density$(X) >$ density(Y). Let L be a "nice" countable fragment of $L_{\omega_1,\omega}[0,1]$. Then there exist Banach spaces X', Y', and a bounded surjective linear map $T' : X' \to Y'$, such that:

- *density$(X') = \aleph_1$,*
- *density$(Y') = \aleph_0$,*
- *$(X, Y, T) \equiv_L (X', Y', T')$.*

This extends a result of Ben Yaacov and Iovino.

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In particular:

- *If X is non-reflexive, we may ensure X' is non-reflexive.*
  - *Similarly for Y and Y'.*
- *If X is not hereditarily indecomposible, we may ensure X' is not hereditarily indecomposible.*
  - *Similarly for Y and Y'.*
- $\cdots$
- *If Y is separable, we may take $Y' = Y$.*

Corollary

If Y is a separable Banach space which is a quotient of a non-separable Banach space X, then Y is also a quotient of a Banach space X' of density $\aleph_1$, with $X \equiv_L X'$.

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Thank you!

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www.math.toronto.edu/cjeagle