The Completeness of Isomorphism

An important theme in DST (Descriptive Set Theory):

Borel reducibility of equivalence relations

If E, F are equivalence relations on the reals then E *is Borel-reducible to* F, $E \leq_B F$, iff

$(*)_B$ There is a Borel (boldface Δ^1_1) total function f (a *Borel reduction*) such that

$$x \; E \; y \leftrightarrow f(x) \; F \; f(y) \text{ for all } x, y$$

Especially important are the analytic (boldface Σ^1_1) equivalence relations, such as isomorphism on countable structures:

$x \simeq y$ iff x, y code countable structures which are isomorphic

An important observation: Isomorphism $\simeq$ on countable structures is *not Borel-complete*:

The Completeness of Isomorphism

Theorem

There are Σ^1_1 equivalence relations which are not Borel-reducible to Isomorphism $\simeq$.

Proof. Let X be a set of reals which is Σ^1_1 but not Borel.
Define: $x \; E_X \; y$ iff $x, y \in X$ or $x = y$

Then E_X is Σ^1_1 and X is a non-Borel equivalence class of E_X.
But:

Theorem

(Scott) The equivalence classes of $\simeq$ are Borel, i.e., if A is a countable structure then the set $[A]_\simeq$ of codes for structures B which are isomorphic to A forms a Borel set.

It follows that E_X cannot Borel-reduce to $\simeq$

The Completeness of Isomorphism

The picture is different in the computable setting.

Suppose E, F are equivalence relations which are effectively Σ^1_1.

E is Hyp-reducible to F on the computable reals iff

$(*)_{Comp}$ There is a Hyp (effectively Borel) total function f on the reals sending computable reals to computable reals such that:

$$x \; E \; y \leftrightarrow f(x) \; F \; f(y) \text{ for all computable } x, y$$

Theorem

(FFHKMM) Every effectively Σ^1_1 equivalence relation is Hyp-reducible to $\simeq$ on the computable reals (i.e., $\simeq$ for computable structures is complete).

Question. For which natural classes of countable structures between the class of computable structures and the class of all countable structures is isomorphism complete?

Classes of structures

Assume $V = L$. We use Gödel's L-hierarchy to define classes of structures as follows:

For a pair (α, n) where α is infinite and $0 < n \in \omega$ define:

$X(\alpha, n)$ = all reals (subsets of ω) which are Δ_n definable over L_α
$S(\alpha, n)$ = all structures on ω with codes in $X(\alpha, n)$

Also when α is a countable ordinal greater than ω define:

$X(\alpha, 0)$ = all reals which are elements of L_α
$S(\alpha, 0)$ = all structures on ω with codes in $X(\alpha, 0)$

Classes of structures

Suppose E, F are equivalence relations on reals which are Σ^1_1 with parameter from $X(\alpha, n)$

E is *Hyp reducible to F on $X(\alpha, n)$* iff there exists a total f on the reals sending $X(\alpha, n)$ into $X(\alpha, n)$ such that for $x, y \in X(\alpha, n)$:

$$x \; F \; y \text{ iff } f(x) \; E \; f(y),$$

where f is Hyp with parameter from $X(\alpha, n)$.

E is *complete on $X(\alpha, n)$* iff every equivalence relation which is Σ^1_1 with parameter from $X(\alpha, n)$ is Hyp reducible to E on $X(\alpha, n)$.

Note that $\simeq$ is a Σ^1_1 equivalence relation without parameter so is a "candidate" for completeness on $X(\alpha, n)$ for each (α, n)

Main Question. For which α, n is $\simeq$ complete on $X(\alpha, n)$?

Reduction to the case $n = 0$

[*Main Question.* For which α, n is $\simeq$ complete on $X(\alpha, n)$?]

We can reduce the problem to the case $n = 0$ using a fine-structural fact:

Theorem

(Δ_n Master Codes) Suppose that $n > 0$ and $X(\alpha, n) \neq X(\alpha, 0)$. Then for some real $c(\alpha, n)$:

$$x \in X(\alpha, n) \text{ iff } x \leq_T c(\alpha, n).$$

Corollary

Suppose that $n > 0$ and $X(\alpha, n) \neq X(\alpha, 0)$. Then $\simeq$ is complete on $X(\alpha, n)$.

Proof. By the FFHKMM Theorem, $\simeq$ is complete on the computable reals. Now relativise to the real $c(\alpha, n)$. $\square$

When α is a limit of admissibles

[*Question.* For which α is $\simeq$ complete on $X(\alpha, 0)$?]

Recall that $\simeq$ is not complete on the set of all reals because of:

Theorem

(Scott) If A is a countable structure then the set $[A]_\simeq$ of codes for structures which are isomorphic to A forms a Borel set.

Refinement: If c is a code for A then $[A]_\simeq$ has a Borel code definable over the least admissible set containing c.

So if c belongs to L_α, α a limit of admissibles then Scott's Theorem holds in L_α and we obtain:

Corollary

If α is a limit of admissibles then $\simeq$ is not complete on $X(\alpha, 0)$.

When α is computable in some real in L_α

[*Question.* For which α, n is $\simeq$ complete on $X(\alpha, n)$?]

Now suppose that α is computable.

Then there is a Hyp bijection between $X(\alpha, 0)$ and the computable reals.

So $\simeq$ is complete on $X(\alpha, 0)$ because it is complete on the computable reals.

By relativisation, if α is computable in some real in L_α then $\simeq$ is complete on $X(\alpha, 0)$.

To summarise, we now have the following:

Reduction to the Hyp Case

Theorem

(1) If $n > 0$ and $X(\alpha, n) \neq X(\alpha, 0)$ then $\simeq$ is complete on $X(\alpha, n)$.
(2) Suppose $X(\alpha, 0) \neq X(\beta, 0)$ for any $\beta < \alpha$. Then:
(a) If α is a limit of admissibles, $\simeq$ is not complete on $X(\alpha, 0)$.
(b) If α neither admissible nor the limit of admissibles, $\simeq$ is complete on $X(\alpha, 0)$.

(The reason for 2(b) is that its hypotheses imply that α is computable in some real in L_α.)

So we are left with the case: α is admissible, not the limit of admissibles and $X(\alpha, 0) \neq X(\beta, 0)$ for $\beta < \alpha$.

This implies that for some real p, $X(\alpha, n)$ is exactly the set of reals Hyp in p. Ignoring p our problem reduces to the following:

The Hyp Case

Key Case. Is $\simeq$ complete on the set of Hyp reals?

I.e., if E is a Σ^1_1 equivalence relation (with Hyp code) is there a total Hyp function f such that for Hyp reals x, y: $x \ E \ y$ iff $f(x), f(y)$ code isomorphic structures?

The method of FFHKMM does not seem to work for the Hyp case: There is no Hyp enumeration of all Hyp reals.

The Scott method does not seem to work either: If A has a Hyp code there need not be a Borel set $\mathcal{B}$ with Hyp code such that $[A]_{\simeq} \cap Hyp = \mathcal{B} \cap Hyp$.

The solution comes from a deeper look at descriptive set theory and infinitary logic.

The Relation E_1

For $x \subseteq \omega$ and $n \in \omega$ define $(x)_n = \{m \mid \langle m, n\rangle \in x\}$, where $\langle ., .\rangle$ is a computable pairing function on ω.

The equivalence relation E_1 is defined by:

$$x \; E_1 y \text{ iff } (x)_n = (y)_n \text{ for large enough } n.$$

E_1 is a Hyp equivalence relation. First we show:

Theorem

Suppose that α is a limit of admissibles. Then E_1 is not reducible to $\simeq$ on $X(\alpha, 0)$: There is no total Hyp function f such that for x, y in L_α, $x \; E_1 \; y$ iff $f(x), f(y)$ code isomorphic structures.

So in fact $\simeq$ is very incomplete on L_α: There are even Hyp equivalence relations which are not Hyp-reducible to $\simeq$ on L_α.

The Relation E_1

[*Theorem.* If α is a limit of admissibles then E_1 is not reducible to $\simeq$ on L_α.]

We outline the proof.

Suppose f were a Hyp-reduction of E_1 to $\simeq$ on L_α.

Define: $\simeq_0 \, = \, \simeq$ and $\simeq_n \, = \, (\simeq$ fixing $0, 1, \ldots, n-1)$.

Choose an admissible $\alpha_0 < \alpha$ so that the code for f belongs to L_{α_0} and α_0 is countable in L_α.

Also write $x \ E_1^k \ y$ iff $x(i) = y(i)$ for $i \geq k$ and $x(i) \restriction k = y(i) \restriction k$ for $i < k$.

The Relation E_1

[*Theorem.* If α is a limit of admissibles then E_1 is not reducible to $\simeq$ on L_α.]

Claim. For each n there is k so that if $g, h \in L_\alpha$ are Cohen-generic over L_{α_0} and $g\ E_1^k\ h$ then $f(g) \simeq_n f(h)$.

Proof Sketch. Let g_0 in L_α be Cohen-generic over L_{α_0} and choose a Cohen condition which forces that $f(g)$ and $f(g_0)$ are isomorphic sending $(0, 1, \ldots, n-1)$ to $\vec{k} = (k_0, k_1, \ldots, k_{n-1})$ for some fixed $\vec{k}$. If g, h in L_α are Cohen-generic over L_{α_0} below this condition then $f(g) \simeq_n f(h)$. $\square$ (*Claim*)

The Relation E_1

Now build a sequence of g^n's in L_α which are Cohen-generic over L_{α_0} so that $g^n \; E_1^{k_n} \; g^{n+1}$ where k_n is large enough to guarantee:

1. $f(g^n) \simeq_{m_n} f(g^{n+1})$ where m_n is large enough so that there is an isomorphism between $f(g^0)$ and $f(g^n)$ under which the images and pre-images of the numbers less than n are all less than m_n.
2. The k_n's go to infinity
3. $g^n(n-1), g^{n+1}(n-1)$ differ somewhere, and
4. $g =$ the limit of the g^n's is Cohen-generic over L_{α_0}.

Then g is not E_1-equivalent to g^0 by 3.
The sequence of g^n's can be built in L_α as α_0 is countable in L_α and any two isomorphic structures in L_α are also isomorphic in L_α.
Using 1, 2 and 4, $f(g^0) \simeq f(g)$.
But this contradicts the assumption that f is a reduction of E_1 to $\simeq$ on L_α. $\square$

The Hyp Case

The difficulty in applying the above argument to the Hyp case is that two Hyp structures can be isomorphic without being Hyp isomorphic.

However this does not happen for Hyp structures of low (computable) Scott rank. So we at least have:

Theorem

There is no Hyp reduction f of E_1 to $\simeq$ on Hyp such that for each Hyp x, $f(x)$ is a structure of low Scott rank.

To complete the argument for Hyp, we use a method for converting arbitrary structures to structures of low Scott rank.

Let $\equiv_\alpha$ denote elementary equivalence for sentences of $L_{\omega_1\omega}$ of rank less than α.

The Hyp Case

Theorem

Suppose that α is a computable ordinal.
Then there is a Hyp function $\mathcal{A} \mapsto \mathcal{A}^*$ on countable structures $\mathcal{A}$ such that:
(a) $\mathcal{A} \simeq \mathcal{B} \to \mathcal{A}^* \simeq \mathcal{B}^*$.
(b) $\mathcal{A}^* \equiv_\alpha \mathcal{B}^* \to \mathcal{A} \equiv_\alpha \mathcal{B}$.
(c) For each $\mathcal{A}$, $\mathcal{A}^*$ has Scott rank at most α.
(In fact, (b) can be made into an equivalence.)

Now if f were a Hyp reduction of E_1 to $\simeq$ on Hyp we could choose a computable α so that f reduces E_1 (on enough of Hyp) to $\equiv_\alpha$. Use the Theorem to ensure that the range of f consists solely of Hyp structures of low Scott rank, which by the previous Theorem yields a contradiction.

Conclusion

Completeness of isomorphism on the $X(\alpha, n)$'s is therefore characterised as follows:

Say that (α, n) is a *relevant pair* if either $n \neq 0$ and $X(\alpha, n) \neq X(\alpha, 0)$ or $X(\alpha, n) = X(\alpha, 0) \neq X(\beta, 0)$ for $\beta < \alpha$.

Clearly only relevant pairs are relevant.

Theorem

Suppose that (α, n) is a relevant pair. Then $\simeq$ is incomplete on $X(\alpha, n)$ iff $n = 0$ and α is either admissible or the limit of admissibles.

And we have seen that if $(\alpha, 0)$ is relevant and α is either admissible or the limit of admissibles then even the Hyp equivalence relation E_1 does not Hyp-reduce to $\simeq$ on $X(\alpha, 0)$.

Questions

But when $(\alpha, 0)$ is relevant and α is a limit of admissibles, one has even more:
E_1 does not reduce to any equivalence relation resulting from a Borel action of a Polish group where both the action and group are coded in L_α.

Is there a Hyp analogue of this result?

Finally, one can ask about the completeness of $\simeq$ on X when X is not of the form $X(\alpha, n)$.
If X is closed under Hyp-reducibility then one obtains the incompleteness of $\simeq$ on X as above.

But what if, for example, $(g_0, g_1, \ldots)$ is a sequence of reals generic for Cohen$^\omega$ over the arithmetical sets and X consists of those reals arithmetical in finitely-many g_i's; is $\simeq$ on X complete?