Recent developments in the model theory of tracial von Neumann algebras

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von Neumann algebras (or what David just said)

- $\mathcal{B}(H)$ denotes the $*$-algebra of all bounded operators on the (complex) Hilbert space.
- For $X \subseteq \mathcal{B}(H)$, $X' := \{T \in \mathcal{B}(H) \ : \ ST = TS \text{ for all } S \in X\}$.
- A *von Neumann algebra* is a $*$-subalgebra M of $\mathcal{B}(H)$ such that $M = M''$.
- A von Neumann algebra is a *factor* if the center of M is trivial (namely $\mathbb{C} \cdot 1$).
- A factor is a II_1 *factor* if it is infinite-dimensional and has a *tracial state*, that is a linear functional $\operatorname{tr} : M \to \mathbb{C}$ such that $\operatorname{tr}(1) = 1$ and, for all $x, y \in M$, $\operatorname{tr}(x^* x) \geq 0$ and $\operatorname{tr}(xy) = \operatorname{tr}(yx)$. (This trace is unique.)
- There are von Neumann algebras that possess a trace that are not II_1 factors (e.g. $M_n(\mathbb{C})$ with its normalized trace). By a *tracial von Neumann algebra*, we will mean a von Neumann algebra equipped with a specific trace.

Tracial von Neumann algebras as metric structures (à la Farah-Hart-Sherman)

- We will treat tracial von Neumann as structures in an appropriate continuous logic for von Neumann algebras.
- There is a theory T_{vna} that axiomatizes the class of tracial von Neumann algebras.
- **Important**: T_{vna} is a universal theory.
- One can also axiomatize the class of II_1 factors by a theory T_{II_1}.
- **Important**: T_{II_1} is $\forall\exists$-axiomatizable.

Existentially closed structures

- If T is a (continuous) theory, $M \models T$ is an *existentially closed (e.c.) model of T* if for any quantifier-free formula $\varphi(x, y)$, any $a \in M$, and any $N \models T$ with $M \subseteq N$, we have

$$(\inf_x \varphi(x, a))^M = (\inf_x \varphi(x, a))^N.$$

- If $M \models T_{vna}$, then there is $N \models T_{II_1}$ such that $M \subseteq N$ (e.g. take $N = M * L(\mathbb{Z})$). In other words, $T_{vna} = (T_{II_1})_\forall$.
- Since T_{II_1} is $\forall\exists$-axiomatizable, it follows that *any e.c. tracial von Neumann algebra is a II_1 factor*.

Ultrapowers of von Neumann algebras

- Suppose that (A, τ) is a tracial von Neumann algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$.
- We set
$$\ell^\infty(A) := \{(a_n) \in A^{\mathbb{N}} \ : \ \|a_n\| \text{ is bounded}\}.$$
- If we quotient out by the ideal
$$\{(a_n) \in A^{\mathbb{N}} \ : \ \lim_{\mathcal{U}} \|a_n\|_2 = 0\},$$
we get the *tracial ultrapower* $A^{\mathcal{U}}$ of A. ($\|x\|_2 := \sqrt{\operatorname{tr}(x^*x)}$.)
- This is precisely the continuous logic ultrapower of A.

$\mathcal{R}^\omega$-embeddability

- Recall that the *hyperfinite II_1 factor* $\mathcal{R}$ is the appropriate completion of $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$.
- $\mathcal{R}$ **embeds into any II_1 factor.** (In model-theoretic lingo: $\mathcal{R}$ is an algebraically prime model of T_{II_1}.)

Definition

We say that a separable II_1 factor A is $\mathcal{R}^\omega$*-embeddable* if there is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that A embeds into $\mathcal{R}^\mathcal{U}$.

Remarks

1. If A is $\mathcal{R}^\omega$-embeddable, then A embeds into $\mathcal{R}^\mathcal{U}$ for *any* nonprincipal ultrafilter on $\mathbb{N}$.
2. A is $\mathcal{R}^\omega$-embeddable if and only if $A \models \mathrm{Th}_\forall(\mathcal{R})$, the universal theory of $\mathcal{R}$.

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1. If A is $\mathcal{R}^\omega$-embeddable, then A embeds into $\mathcal{R}^\mathcal{U}$ for *any* nonprincipal ultrafilter on $\mathbb{N}$.
2. A is $\mathcal{R}^\omega$-embeddable if and only if $A \models \mathrm{Th}_\forall(\mathcal{R})$, the universal theory of $\mathcal{R}$.

Connes' Embedding Problem

- In 1976, Connes proved that $L(\mathbb{F}_2)$ is $\mathcal{R}^\omega$-embeddable.
- He then remarked "Apparently such an embedding ought to exist for all (separable) II_1 factors..."
- This remark is now known as the *Connes Embedding Problem* (CEP) and is *the* central question in operator algebras. It has zillions of equivalent reformulations.
- For example, it is known that $L(G)$ is $\mathcal{R}^\omega$-embeddable if and only if G is hyperlinear. So CEP for groups is equivalent to the statement that every group is hyperlinear.
- One can replace the words "II_1 factor" with "tracial von Neumann algebra."

CEP in model-theoretic lingo

- CEP: Every separable II_1 factor embeds into $\mathcal{R}^{\mathcal{U}}$.
- So CEP is equivalent to $\mathrm{Th}_\forall(\mathcal{R}) = (T_{II_1})_\forall$.
- Call a separable II_1 factor A *locally universal* if every separable II_1 factor is A^ω-embeddable, that is, if $\mathrm{Th}_\forall(A) \subseteq (T_{II_1})_\forall$.
- CEP asks whether or not $\mathcal{R}$ is locally universal.
- Hart, Farah, and Sherman proved the existence of one (and therefore many) locally universal II_1 factors ("Poor man's CEP").

Model companions

- Recall that a theory T is *model complete* if any embedding between models of T is elementary.
- If T' is a theory, then a model complete theory T is a *model companion* for T' if any model of T' embeds in a model of T and vice-versa (that is, if $T'_\forall = T_\forall$). A theory can have at most one model companion.
- If T' is universal, then T' has a model companion T if and only if the class of existentially closed models of T' is elementary; in this case T is their theory.

Theorem (G., Hart, Sinclair)

T_{vNa} does not have a model companion.

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Theorem (G., Hart, Sinclair)

T_{vNa} does not have a model companion.

$\mathcal{R}$ does not have QE

Theorem (GHS)

$\mathcal{R}$ does not have QE.

Proof.

- It is enough to find $\mathcal{R}^\omega$-embeddable von Neumann algebras M and N with $M \subset N$ and an embedding $\pi : M \hookrightarrow \mathcal{R}^\mathcal{U}$ that does not extend to an embedding $N \hookrightarrow \mathcal{R}^\mathcal{U}$.
- Towards this end, it is enough to find a countable group G such that $L(G)$ is $\mathcal{R}^\omega$-embeddable, an embedding $\pi : L(G) \hookrightarrow \mathcal{R}^\mathcal{U}$, and $\alpha \in \mathrm{Aut}(L(G))$ such that there exists no unitary $u \in \mathcal{R}^\mathcal{U}$ satisfying $(\pi \circ \alpha)(x) = u\pi(x)u^*$ for all $x \in L(G)$. (We'll explain this on the next slide.)
- By nontrivial work of Nate Brown, we can take $G = \mathrm{SL}(3, \mathbb{Z}) * \mathbb{Z}$ and $\alpha = \mathrm{id} * \theta$ for any nontrivial $\theta \in \mathrm{Aut}(L(\mathbb{Z}))$.

$\mathcal{R}$ does not have QE (cont'd)

Proof.

- Suppose that G, π, and α are as above. Set $M := L(G)$ and $N := M \rtimes_\alpha \mathbb{Z}$. Then N is $\mathcal{R}^\omega$-embeddable.
- Suppose, towards a contradiction, that π extends to $\tilde{\pi} : N \hookrightarrow \mathcal{R}^\mathcal{U}$.
- Let $u \in N$ be the generator of $\mathbb{Z}$ and set $\tilde{u} := \tilde{\pi}(u)$. We then have, for $x \in M$:

$$\tilde{u}\pi(x)\tilde{u}^* = \tilde{\pi}(uxu^*) = \pi(\alpha(x)),$$

contradicting our choice of π and α. □

Other non-QE results

Definition

If A is a separable II_1 factor, we say that A is *McDuff* if $A \otimes \mathcal{R} \cong A$.

- For example, $\mathcal{R}$ is McDuff.
- Any II_1 factor A embeds into a McDuff factor: $A \subseteq A \otimes \mathcal{R}$.
- It is a fact that McDuffness is $\forall\exists$-axiomatizable, whence a separable existentially closed tracial von Neumann algebra is a McDuff II_1 factor.

We noticed that Brown's work would apply if instead of $\mathcal{R}$ we had a locally universal, McDuff II_1 factor. We thus have:

Theorem (GHS)

If $\mathcal{S}$ is a locally universal, McDuff II_1 factor, then $\mathcal{S}$ does not have QE.

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Proof of the Main Theorem

- Suppose, towards a contradiction, that T_{vNa} has a model companion T. Since T_{vNA} is $\forall$-axiomatizable and has the amalgamation property, we have that T has QE.
- Fix a separable model $\mathcal{S}$ of T. As discussed earlier, models of T are then existentially closed tracial von Neumann algebras, whence $\mathcal{S}$ is a McDuff II_1 factor.
- Moreover, $\mathcal{S}$ is a locally universal II_1 factor: if A is an arbitrary separable tracial vNA, then A embeds in some separable $\mathcal{S}_1 \models T$. Since $\mathcal{S}^{\mathcal{U}}$ is ω_1-saturated, $\mathcal{S}_1$ embeds in $\mathcal{S}^{\mathcal{U}}$, whence A embeds in $\mathcal{S}^{\mathcal{U}}$.
- By our previous theorem, $\mathcal{S}$ does not have QE, a contradiction. □

Are there model complete theories of tracial vNas?

Just because there is no model companion of T_{vNA} does not prevent there from being a model-complete theory of tracial von Neumann algebras, so we raise the question: Is there a model-complete theory of tracial von Neumann algebras (whose models would automatically be II_1 factors)?

Theorem (G., Hart, Sinclair)

If the CEP has a positive solution, then there is no model-complete theory of tracial von Neumann algebras.

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Theorem (G., Hart, Sinclair)

If the CEP has a positive solution, then there is no model-complete theory of tracial von Neumann algebras.

A preliminary result

Fact (Jung? von Neumann?)

Any embedding $\mathcal{R} \to \mathcal{R}^{\mathcal{U}}$ is unitarily equivalent to the diagonal embedding (whence elementary).

Remark

The previous result shows that $\mathcal{R}$ is the prime model of its theory.

$\mathcal{R}$ is the only possibility

Proposition (GHS)

Suppose that A is an $\mathcal{R}^\omega$-embeddable II_1 factor such that Th(A) is $\forall\exists$-axiomatizable. Then $A \equiv \mathcal{R}$.

Proof.

Draw crude diagram on the board. □

We now see how CEP implies that there is no model-complete theory of II_1 factors. Indeed, if T were a model-complete theory of II_1 factors, then by CEP, models of T would be $\mathcal{R}^\omega$-embeddable, whence the above proposition shows $T = \mathrm{Th}(\mathcal{R})$. Another use of CEP shows that $T_\forall = T_{vNa}$, whence T is a model companion for T_{vNa}, which we know has no model companion.

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What about $\mathrm{Th}_\forall(\mathcal{R})$?

- Suppose that CEP is false, so that $T_{vNA} \neq \mathrm{Th}_\forall(\mathcal{R})$.
- Could $\mathrm{Th}_\forall(\mathcal{R})$ have a model companion?

Corollary

Suppose that CEP fails but that $\mathrm{Th}_\forall(\mathcal{R})$ has the amalgamation property. Then $\mathrm{Th}_\forall(\mathcal{R})$ does not have a model companion.

Proof.

The model companion would have to be $\mathrm{Th}(\mathcal{R})$ by the last slide and would have to have quantifier elimination by assumption. □

Corollary

If $\mathrm{Th}(\mathcal{R})$ is model complete, then $\mathrm{Th}_\forall(\mathcal{R})$ does not have the amalgamation property.

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Corollary

If $\mathrm{Th}(\mathcal{R})$ *is model complete, then* $\mathrm{Th}_\forall(\mathcal{R})$ *does not have the amalgamation property.*

e.c. models of $\mathrm{Th}_\forall(\mathcal{R})$

- The fact that every embedding $\mathcal{R} \to \mathcal{R}^\mathcal{U}$ is elementary can be used to show that $\mathcal{R}$ *is an e.c. model of* $\mathrm{Th}_\forall(\mathcal{R})$.
- Can we name any other e.c. models of $\mathrm{Th}_\forall(\mathcal{R})$?
- How about our favorite $\mathcal{R}^\omega$-embeddable II_1 factors $\prod_\mathcal{U} M_n(\mathbb{C})$ or $L(\mathbb{F}_2)$?
- If $M \models \mathrm{Th}_\forall(\mathcal{R})$ is e.c., then $\mathrm{Th}_{\forall\exists}(\mathcal{R}) \subseteq \mathrm{Th}_{\forall\exists}(M)$.

Corollary

$\prod_\mathcal{U} M_n(\mathbb{C})$ and $L(\mathbb{F}_2)$ are not e.c.

Proof.

Property (Γ). □

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Digression: a wager

We saw earlier the following fact:

Lemma

The only possible complete $\forall\exists$-axiomatizable theory of $\mathcal{R}^\omega$-embeddable II_1 factors is $\mathrm{Th}(\mathcal{R})$. In particular, $\mathrm{Th}(\prod_{\mathcal{U}} M_n(\mathbb{C}))$ and $\mathrm{Th}(L(\mathbb{F}_2))$ are not $\forall\exists$-axiomatizable.

A wager

- If $\mathrm{Th}(\mathcal{R})$ is $\forall\exists$-axiomatizable, then Bradd buys me a drink.
- If $\mathrm{Th}(\mathcal{R})$ is not $\forall\exists$-axiomatizable, then I buy Bradd a drink.
- Somehow, either way, Ilijas gets a drink!

We know that $\mathrm{Th}(\mathcal{R})$ is not $\exists\forall$-axiomatizable: $\mathcal{R} \hookrightarrow L(\mathbb{F}_2) \hookrightarrow \mathcal{R}^{\mathcal{U}}$.

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Companion operators

We can view the map $T \mapsto T^*$ (where T^* is the model companion of T) as a partial map on theories. Companion operators are total extensions of this map.

Definition

Suppose that we have a mapping $T \mapsto T^*$ on theories. We say that the mapping is a *companion operator* if, for all theories T and T', we have:

1. $(T^*)_\forall = T_\forall$
2. $T_\forall = T'_\forall \Rightarrow T^* = (T')^*$.
3. $T_{\forall\exists} \subseteq T^*$.

If $*$ is a companion operator and the model companion of T exists, then it is T^*.

Kaiser hull

- There is a smallest companion of a theory T, its *Kaiser hull* T^{KH}.
- It is the largest $\forall\exists$-axiomatizable theory whose universal theory is the same as the universal theory of T.
- If T is universal, then T^{KH} is axiomatized by $\mathrm{Th}_{\forall\exists}(M)$ for any e.c. $M \models T$.

Corollary

If $T = \mathrm{Th}_\forall(\mathcal{R})$*, then* T^{KH} *is axiomatized by* $\mathrm{Th}_{\forall\exists}(\mathcal{R})$*.*

Finite forcing

- Work in $L(C) := L \cup C$, C countably many new constant symbols.
- Conditions: finite sets of atomic/negated atomic $L(C)$-sentences.
- Notion of $p \Vdash \varphi$, φ an $L(C)$-sentence.
- Notion of generic set G of conditions: for every φ, either $G \Vdash \varphi$ or $G \Vdash \neg\varphi$.
- Generic model: $C/\sim_G$, where $c \sim_G d$ iff $G \Vdash c = d$.
- An L-structure M is *finitely generic* if M is isomorphic to the L-reduct of a generic model. Finitely generic structures are existentially closed.
- The common theory of the finitely generic models of T is T^f and is a companion of T, called the *finite forcing companion of T*.
- If T has JEP, then T^f is complete.

Finite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$

Lemma

If $N \preceq_1 M$ and M is a finitely generic model of T, then N is a finitely generic model of T.

Corollary

$\mathcal{R}$ is finitely generic and $\mathrm{Th}(R)$ is the finite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$. ("$\mathrm{Th}(\mathcal{R})$ is complete for finite forcing.")

Proof.

Let $M \models \mathrm{Th}_\forall(\mathcal{R})$ be finitely generic. Since $\mathcal{R}$ embeds into M and is e.c., we have that $\mathcal{R} \preceq_1 M$. □

Infinite forcing

- One defines what it means for a structure M to force an $L(M)$-sentence σ: $M \Vdash \sigma$.
- Key clause: $M \Vdash \neg\sigma$ if and only if there does not exist $N \supseteq M$ such that $N \Vdash \sigma$.
- M is *infinitely generic* if and only if, for every $L(M)$-sentence σ, we have $M \Vdash \sigma$ or $M \Vdash \neg\sigma$. (Equivalently: $M \Vdash \sigma$ if and only if $M \models \sigma$.)
- Infinitely generic structures are e.c. (They are in a sense "super e.c.")
- The common theory of the infinitely generic models of T is T^F and is a companion of T, called the *infinite forcing companion of T.*
- If T has JEP, then T^F is complete.

Infinite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$

Lemma

If $\mathrm{Th}_\forall(\mathcal{R})$ is $\forall\exists$-axiomatizable, then $\mathcal{R}$ is infinitely generic and $\mathrm{Th}(\mathcal{R})$ is the infinite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$.

Proof.

If $\mathrm{Th}(\mathcal{R})$ is the infinite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$, then $\mathcal{R}$ is infinitely generic since $\mathcal{R}$ is the prime model of its theory and an elementary substructure of an infinitely generic structure is infinitely generic. $\mathrm{Th}(\mathcal{R}) = \mathrm{Th}_\forall(\mathcal{R})^F$ is infinitely generic: crude diagram again. □

Is Bradd winning?

If $\mathrm{Th}(\mathcal{R})$ is $\forall\exists$-axiomatizable, then $\mathcal{R}$ is both finitely generic and infinitely generic.

Infinite forcing companion of $\mathrm{Th}_\forall(\mathcal{R})$

Lemma

If $\mathrm{Th}_\forall(\mathcal{R})$ *is* $\forall\exists$*-axiomatizable, then* $\mathcal{R}$ *is infinitely generic and* $\mathrm{Th}(\mathcal{R})$ *is the infinite forcing companion of* $\mathrm{Th}_\forall(\mathcal{R})$*.*

Proof.

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$\mathrm{Th}(\mathcal{R}) = \mathrm{Th}_\forall(\mathcal{R})^F$ is infinitely generic: crude diagram again. □

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Amalgamating

- Recall the classical fact that an e.c. model of a theory T is a *strong amalgamation base* for T.
- This yields a whole slew (=$2^{\aleph_0}$) of $\mathcal{R}^\omega$-embeddable von Neumann algebras that we can (strongly) amalgamate over.
- In particular, $\mathcal{R}$ is a strong amalgamation base.
- Compare this with the difficult fact (due to Brown-Dykema-Jung) that if M_1 and M_2 are $\mathcal{R}^\omega$-embeddable, so is $M_1 *_{\mathcal{R}} M_2$.

CEP and Model-Completeness Revisited

Proposition (Farah, G., Hart)

Suppose that M as in the proof of failure of QE is an *amalgamation base* for the class of $\mathcal{R}^\omega$-embeddable von Neumann algebras. Then $\mathrm{Th}(\mathcal{R})$ is not model complete.

Proof.

Yet another crude diagram. □

- So proving that $\mathrm{Th}(\mathcal{R})$ is model-complete would identify a specific counterexample to the amalgamation property. (Bradd screaming in my ear: What are the chances of that?!?)

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