Unitary equivalence of automorphisms of C*-algebras

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A representation of Γ on $\mathcal{H}$ can be regarded as
an **action of** Γ on $\mathcal{H}$ **by unitary linear transformations**.

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Problem:
classify of the irreducible representations of a group Γ up to equivalence.

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Is it possible to extend such classification to the case of infinite groups?

Abelian groups

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This offers a concrete (smooth) classification of representations of Γ.

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The quotient space $Irr(\Gamma, \mathcal{H}) / \approx$ of $Irr(\Gamma, \mathcal{H})$ modulo the relation $\approx$ of unitary equivalence parametrizes the equivalence classes of representations of dimension $\dim \mathcal{H}$.

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What happens when $\mathcal{H}$ is infinite-dimensional?

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In particular, if there is at least one infinite-dimensional representation, then the quotient topology does not provide a nice classifying space for infinite-dimensional representations.

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3. *$Irr(\Gamma, \mathcal{H}) / \approx$ is standard for $\mathcal{H}$ infinite dimensional*
4. *the relation $\approx_\Gamma$ of equivalence of representations of Γ is smooth*

Dichotomy

Beyond the type I case, one might hope to classify the representations of a group Γ using some class of countable structures as **invariants**.

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The proof of such result uses Hjorth's theory of turbulence.

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The proof of such result uses Hjorth's theory of turbulence.
Hjorth's theorem shows a dichotomy in the Borel complexity.

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If $G \curvearrowright X$ is turbulent, then the associated orbit equivalence relation is **not classifiable by countable structures***.*
It is in fact even F-ergodic for every equivalence relation F that is classifiable by countable structures.

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Problem

Are there Γ, Γ' non type I such that $\approx_\Gamma$ and $\approx_{\Gamma'}$ are not bi-reducible?

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Moreover every commutative C*-algebra is of this form (Gelfand-Naimark).

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The direct union

$$\bigcup_k \mathbb{M}_{2^k}$$

is a *-algebra whose completion

$$\mathbb{M}_{2^\infty}$$

is a C*-algebra called CAR algebra.

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Glimm's result in particular shows that

$$\approx_{\mathbb{M}_{2^\infty}} \leq \approx_A$$

for any other non type I C*-algebra A.

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Kerr-Li-Pichot proof prove directly (generic) turbulence for the Polish group action that has $\approx_A$ as orbit equivalence relation.

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This problem is equivalent to the corresponding one for groups.

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The group $\mathrm{Aut}\,(A)$ of automorphisms of A is Polish with respect to the **topology of pointwise norm convergence**.

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The group $\mathrm{Inn}(A)$ of inner automorphisms is a **Borel normal subgroup** of $\mathrm{Aut}(A)$.

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John Phillips characterized such C*-algebras.

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Theorem (J. Phillips, 1985)

$\mathrm{Inn}(A)$ *is closed in* $\mathrm{Aut}(A)$ *iff* A *is locally homogeneous*

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More generally the same is true when A is a strongly self absorbing C*-algebra (such as $\mathbb{M}_{n^\infty}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$, $\mathcal{Z}$,...).

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