On a question of Abraham and Cummings

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Introduction

Definition

The notation $\omega_2 \nrightarrow_{poly} (\omega_1)^2_{2-bd}$ means that there is a function f such that:

(*a*) $dom(f) = [\omega_2]^2$,

(*b*) f is 2–bounded (the preimage of any elem. of the range of f has size at most 2), and

(*c*) f does not have rainbows of order–type ω_1 ($Y \subseteq \omega_2$ is a rainbow iff the restriction of f to $[Y]^2$ is one to one).

Theorem

(CH) *If κ is such that $\kappa \geq \omega_2$ and $\kappa^{<\kappa} = \kappa$, then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition such that* $\mathrm{MA}(\mathbb{B})$, $2^{\aleph_0} = 2^{\aleph_1} = \kappa$ *and* $\omega_2 \nrightarrow_{poly} (\omega_1)^2_{2-bd}$ *hold in* $V^{\mathcal{P}}$.

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Definition (Baumgartner Forcing for adding a club with finite conditions)

A condition of $\mathbb{B}$ is a finite functions f such that there is a normal function $F : \omega_1 \longrightarrow \omega_1$ including f and $F(\omega) = \omega$. Given conditions f and g, we say that $g \leq_{\mathbb{B}} f$ in case f is included in g.

Notation: If $N \cap \omega_1 \in OR$, then $\delta_N := N \cap \omega_1$.

Remark

If N is a countable elementary substructure of $H(\omega_2)$, $f \in \mathbb{B} \cap N$, then $f \cup \{(N \cap \omega_1, N \cap \omega_1)\}$ is $(N, \mathbb{B})$–generic.

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(a) $\text{MA}_\kappa(\mathbb{B})$ states that for every collection $\{A_\alpha : \alpha \in \kappa\}$ of maximal antichains of $\mathbb{B}$, there is a filter $G \subseteq \mathbb{B}$ meeting each of them.

(b) $\text{MA}(\mathbb{B})$ states that $\text{MA}_\kappa(\mathbb{B})$ is true for every $\kappa < 2^{\omega_1}$.

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Definition (Symmetric Systems)

Let $P \subseteq H(\kappa)$, and let $\{N_i : i < m\}$ be a finite set of countable subsets of $H(\kappa)$. $\{N_i : i < m\}$ *is a P–symmetric system* if

(*A*) For every $i < m$, $(N_i, \in, P) \prec (H(\kappa), \in, P)$.

(*B*) Given i, i' in m, if $\delta_{N_i} = \delta_{N_{i'}}$, then there is a (unique) isomorphism $\Psi_{N_i, N_{i'}} : (N_i, \in, P) \longrightarrow (N_{i'}, \in, P)$ being the identity on $N_i \cap N_{i'}$.

(*C*) For all i, j in m, if $\delta_{N_j} < \delta_{N_i}$, then there is some $i' < m$ such that $\delta_{N_{i'}} = \delta_{N_i}$ and $N_j \in N_{i'}$.

(*D*) For all i, i', j in m, if $N_j \in N_i$ and $\delta_{N_i} = \delta_{N_{i'}}$, then there is some $j' < m$ such that $\Psi_{N_i, N_{i'}}(N_j) = N_{j'}$.

This general presentation of symmetric systems (in the context of $(H(\kappa), \in, P)$ and not only of $(H(\omega_2), \in)$) and the following results may be found in the work of Asperó-Mota.

Definition

The poset $\mathcal{S}_P$ is the set of all the P–symmetric systems Given s_1 and s_0 in $\mathcal{S}_P$, $s_1 \leq_{\mathcal{S}_P} s_0$ iff $s_0 \subseteq s_1$.

Lemma

Let $P \subseteq H(\kappa)$, let $\mathcal{N}$ be a P–symmetric system, and let $N \in \mathcal{N}$. Then:

(i) *$\mathcal{N} \cap N$ is also a P–symmetric system.*

(ii) *If $\mathcal{W} \subseteq N$ is a P–symmetric system and $\mathcal{N} \cap N \subseteq \mathcal{W}$, then*

$$\mathcal{V} := \mathcal{N} \cup \{\Psi_{N,N'}(W) \,:\, W \in \mathcal{W},\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

is a P–symmetric system.

Corollary

$\mathcal{S}_P$ is proper. Moreover, suppose that θ is a regular cardinal and N^* is a ctble. elem. substructure of $H(\theta)$ such that $\mathcal{S}_P \in N^*$. Then, letting $N = N^* \cap H(\kappa)$, the following hold.

1. *For every $s \in N \cap \mathcal{S}_P$, $s' \cup \{N\} \leq_{\mathcal{S}_P} s$.*
2. *If $N \in s$, then s is $(N^*, \mathcal{S}_P)$–generic.*

Proof.

Let E be a dense subset of $\mathcal{S}_P$ in N^*. It suffices to show that there is some condition in $E \cap N^*$ compatible with s. Notice that $s \cap N^* \in \mathcal{S}_P$ by (i) in the above lemma. Hence, we may find a condition $s^\circ \in E \cap N^*$ extending $s \cap N^*$. Now let

$$s^* = s \cup \{\Psi_{N,\overline{N}}(M) \,:\, M \in q^\circ, \overline{N} \in s, \delta_{\overline{N}} = \delta_N\}$$

By (ii), s^* is a condition in $\mathcal{S}_P$ extending both s and s°. $\square$

Lemma

Let $P \subseteq H(\kappa)$ and let $\mathcal{N}_0 = \{N_i^0 : i < m\}$ and $\mathcal{N}_1 = \{N_i^1 : i < m\}$ be P–symmetric systems. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = X$ and that there is an isomorphism Ψ between the structures $\langle \bigcup_{i<m} N_i^0, \in, P, X, N_i^0 \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^1, \in, P, X, N_i^1 \rangle_{i<m}$ fixing X. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a P–symmetric system.

Lemma

(CH) If there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, then $\mathcal{S}_P$ is $\aleph_2$–Knaster.

Proof. Suppose that $s_\xi = \{N_i^\xi : i < m\}$ is a $\mathcal{S}_P$–condition for each $\xi < \omega_2$. By CH we may assume that $\{\bigcup_{i<m} N_i^\xi : \xi < \omega_2\}$ forms a Δ–system with root X. Moreover, by CH we may assume, for all $\xi, \xi' < \omega_2$, that the structures $\langle \bigcup_{i<m} N_i^\xi, \in, P, X, N_i^\xi \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, P, X, N_i^{\xi'} \rangle_{i<m}$ are isomorphic and that the isomorphism fixes X.

The first assertion follows from the fact that there are only $\aleph_1$–many iso. types for such structures. For the second assertion note that, if Ψ is the unique isomorphism between $\langle\bigcup_{i<m} N_i^\xi, \in, P, X, N_i^\xi\rangle_{i<m}$ and $\langle\bigcup_{i<m} N_i^{\xi'}, \in, P, X, N_i^{\xi'}\rangle_{i<m}$, then the restriction of Ψ to $X \cap \kappa$ has to be the identity on $X \cap \kappa$. Since there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, we have that Ψ fixes X if and only if it fixes $X \cap \kappa$. It follows that Ψ fixes X. Hence, for all ξ, $\xi' < \omega_2$, $s_\xi \cup s_{\xi'}$ extends both s_ξ and $s_{\xi'}$.

Corollary

(CH) If there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, then $\mathcal{S}_P$ preserves CH.

Proof. Suppose $s \in \mathcal{S}_P$, $\dot{r}_\alpha$ (for $\alpha < \omega_2$) are $\mathcal{S}_P$–names , and s forces that $\dot{r}_\alpha$, for $\alpha < \omega_2$, are pairwise distinct reals . By the $\aleph_2$–chain condition of $\mathcal{S}_P$ we may assume that each $\dot{r}_\alpha$ is in $H(\kappa)$. Let θ be a regular cardinal such that $\mathcal{S}_P \in H(\theta)$. For each α let N_α be such that $\{q, \dot{r}_\alpha\} \in N_\alpha$ and N_α is a countable elementary substructure of $(H(\kappa), \in, P, \mathcal{S}_P)$. We can also assume that for each α, there is a countable elementary substructure $N^*_\alpha \prec H(\theta)$ such that $N_\alpha = H(\kappa) \cap N^*_\alpha$. By CH, there are distinct α, α' such that $(N_\alpha, \in, P, \mathcal{S}_P, s, \dot{r}_\alpha)$ and $(N_\alpha, \in, P, \mathcal{S}_P, s, \dot{r}_\alpha)$ are isomorphic.

By the above two lemmas we may also assume that $s \cup \{N_\alpha, N_{\alpha'}\}$ is a $\mathcal{S}_P$–condition. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ is $(N^*_\alpha, \mathcal{S}_P)$–generic and $(N^*_{\alpha'}, \mathcal{S}_P)$–generic. Let Ψ be the unique isomorphism between N_α and $N_{\alpha'}$ and note that for every natural number n and for every condition s' $\mathcal{S}_P$–extending $s \cup \{N_\alpha, N_{\alpha'}\}$, there are conditions s'' and r such that $r \in N_\alpha$, r decides the nth value of $\dot{r}_\alpha$ and s'' is a common $\mathcal{S}_P$–extension of r and s'. Since symmetric systems are closed under isomorphism, s'' also $\mathcal{S}_P$–extends $\Psi(r) \in N_{\alpha'}$. By correctness of Ψ, $\Psi(r)$ forces that the nth value of $\Psi(\dot{r}_\alpha) = \dot{r}_{\alpha'}$ is equal to the nth value of $\dot{r}_\alpha$. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ forces that $\dot{r}_\alpha = \dot{r}_{\alpha'}$. This is a contradiction.

Proposition

Let κ be a regular cardinal such that $\kappa \geq \omega_3$. If there is a bijection $\varphi : H(\kappa) \longrightarrow \kappa$ definable in $(H(\kappa), \in, P)$, then there is, in $V^{\mathcal{S}_P}$, a set $\{X^\alpha_\delta : \alpha < \omega_2, \delta < \omega_1\}$ such that:

1. *X^α_δ is a countable subset of α,*
2. *$\alpha = \bigcup\{X^\alpha_\delta : \delta < \omega_1\}$, and*
3. *If $\alpha \in X^{\alpha'}_\delta$, then $X^\alpha_\delta = X^{\alpha'}_\delta \cap \alpha$.*

Proof.

Let $\langle f_\alpha : \alpha < \omega_2\rangle$ be the φ–least sequence such that each f_α is a surjective function from ω_1 to α. For each $\alpha \in \omega_2$ and for each countable ordinal σ, define D^α_σ as the set of of those conditions $s \in \mathcal{S}_P$ such that there is $N \in s$ with $\alpha \in N$ and $\delta_N \geq \sigma$. Note that D^α_σ is a dense subset of $\mathcal{S}_P$. So, if G is a $\mathcal{S}_P$–generic filter, then we can define X^α_δ as $N \cap \sigma$, where N is any element of G such that $\delta_N = \delta$ and $\alpha \in N$ (note that in such a case $N \cap \alpha$ is equal to the image of α by f_δ). Otherwise, define $X^\alpha_\delta = \emptyset$. □

Abraham and Cummings forcing construction

Using symmetric systems, Abraham and Cummings generically add a function c such that:

(1) $dom(c) = [\omega_2]^2$,

(2) There are no $\alpha_0 < \alpha_1 < \alpha_2 < \beta$ such that $c(\alpha_0, \beta) = c(\alpha_1, \beta) = c(\alpha_2, \beta)$, and

(3) For every $X \subseteq \omega_2$ with order–type equal to ω_1, there are $\alpha_0 < \alpha_1 < \beta$ (all in X) such that $c(\alpha_0, \beta) = c(\alpha_1, \beta)$.

It is straightforward to check that $f(\alpha, \beta) = (c(\alpha, \beta), \beta)$ witnesses $\omega_2 \not\rightarrow_{poly} (\omega_1)^2_{2-bd}$.

In the following, we will describe the forcing adding c. From now on we will assume the following cardinal assumptions: CH, $\kappa \geq \omega_2$ and $\kappa^{<\kappa} = \kappa$. We also fix a surjection $\Phi : \kappa \longrightarrow H(\kappa)$.

Definition

Conditions in $\mathcal{P}_1$ are pairs $p = (c_p, \Delta_p)$ such that:

(a1) c_p is a finite partial function from $[\omega_2]^2$ to ω_1,

(a2) There are no $\alpha_0 < \alpha_1 < \alpha_2 < \beta$ such that $c(\alpha_0, \beta) = c(\alpha_1, \beta) = c(\alpha_2, \beta)$, and

(a3) For every pair (α, β) in the domain of c_p, $c_p(\alpha, \beta) \geq f_\beta(\alpha)$, where f_β is the Φ–least bijection between β and $|\beta|$.

(a4) Δ_p is a finite set of pairs $\{(N_1, 1), \ldots, (N_n, 1)\}$ such that $\{N_1, \ldots, N_n\}$ is a Φ–symmetric system.

Given p and q conditions in $\mathcal{P}_1$, we will say that $q \leq_1 p$ whenever the following conditions hold:

(a5) $c_p \subseteq c_q$,

(a6) $\Delta_p \subseteq \Delta_q$, and

(a7) For every pair $(\alpha, \beta) \in dom(c_q) \setminus dom(c_p)$ and every pair $(N, 1) \in \Delta_p$, if $(\alpha, \beta) \in N$, then $c_q(\alpha, \beta) \in N$.

The proofs of the following three lemmata can be found in the paper of Abraham-Cummings.

Lemma (First antirainbow lemma of A-C)

Let $p = (c_p, \Delta_p)$ and let $\alpha_0 \leq \alpha_1 < \beta < \omega_2$ be such that:

(1) (α_0, β) *and* (α_1, β) *are not in the domain of* c_p*, and*

(2) *For every N in the domain of* Δ_p*,* (α_0, β) *is in N iff* (α_1, β) *is in N,*

Then there exists $q = (c_q, \Delta_q) \leq_1 p$ *such that* $c_q(\alpha_0, \beta) = c_q(\alpha_1, \beta)$*.*

Lemma

$\mathcal{P}_1$ *has the* $\aleph_2$*–chain condition.*

Lemma (Lemma of properness of A-C)

Let θ be a sufficiently large regular cardinal, $p = (c_p, \Delta_p)$ an element of $\mathcal{P}_1$ and N^* a countable elementary substructure of $H(\theta)$ such that p, $\mathcal{P}_1$ and Φ are in N^*. Let $N = N^* \cap H(\kappa)$ and let $p^+ = (c_p, \Delta_p \cup \{(N, 1)\})$. Then:

(1) $p^+ \leq_1 p$,

(2) *Assuming $q = (c_q, \Delta_q) \leq_1 p^+$ and letting $c_r = c \restriction dom(c_q) \cap N$, $\Delta_r = \Delta_q \cap N$ and $r = (c_r, \Delta_r)$, the following are true:*

  (2.1) *r is an element of $N \cap P_1$,*

  (2.2) *If $s \in N \cap P_1$, $s \leq_1 r$, $q' = (c_{q'}, \Delta_{q'}) \leq_1 q = (c_q, \Delta_q)$ and $c_r = c_{q'} \restriction dom(c_{q'}) \cap M$, then q' and s are $\leq_1$–compatible.*

(3) *p^+ is an $(N^*, \mathcal{P}_1)$–generic condition.*

Abraham-Cummings forcing in the context of Asperó-Mota iteration

The forcing $\mathcal{P}$ witnessing our theorem is the poset $(\mathcal{P}_\kappa, \leq_\kappa)$, where $\langle(\mathcal{P}_\alpha, \leq_\alpha) : 1 \leq \alpha \leq \kappa\rangle$ is defined as follows. $(\mathcal{P}_1, \leq_1)$ has the same definition as above. In general, if $1 \leq \alpha \leq \kappa$, then all the elements of $\mathcal{P}_\alpha$ will be defined as ordered pairs.

Notation: If q is an ordered pair, we denote the first component of q by F_q and the second component of q by Δ_q. Also, if q is an ordered pair such that F_q is a function and Δ_q is a relation and ξ is an ordinal, *the restriction of q to ξ*, denoted by $q|_\xi$, is defined as the pair

$$q|_\xi := (F_q \restriction \xi, \{(N, min\{\beta, \xi\}) : (N, \beta) \in \Delta_q\})$$

Let $\langle \theta_\alpha : 1 \leq \alpha \leq \kappa \rangle$ be the strictly increasing seq. of regular cardinals defined as $\theta_1 = |2^\kappa|^+$ and $\theta_\alpha = |2^{sup\{\theta_\beta : \beta \leq \alpha\}}|^+$ if $\alpha > 1$. For each $\alpha \leq \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all ctble. elem. substructures of $H(\theta_\alpha)$ containing Φ and $\langle \theta_\beta : \beta < \alpha \rangle$. Let also $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}^*_\alpha\}$ and note that if $\alpha < \beta$, then $\mathcal{M}^*_\alpha$ belongs to all members of $\mathcal{M}^*_\beta$ containing α.

If $1 < \alpha \leq \kappa$, the definition of $\mathcal{P}_\alpha$ is as follows. Conditions in $\mathcal{P}_\alpha$ are pairs of the form $q = (F_q, \Delta_q)$ with the following properties.

(b0) F_q is a finite function with $dom(F_q) \subseteq \alpha$.

(b1) Δ_q is of the form $\{(N_i, \beta_i) : i < m\}$ where, for all $i < m$, $\beta_i \leq \alpha \cap sup(N_i \cap \kappa)$.

(b2) The restriction of q to 1 is a condition in $\mathcal{P}_1$.

(b3) If $\xi \in dom(F_q)$ and $\xi \geq 1$, then $F_q(\xi) \in \mathbb{B}$.

(b4) If $\xi \in dom(F_q)$, $\xi \geq 1$, $(N, \beta) \in \Delta_q$, $\beta \geq \xi + 1$, and $\xi \in N$, then δ_N is a fixed point of $F_q(\xi)$.

Given conditions

$$q^{\epsilon} = (F_{\epsilon}, \{(N_i^{\epsilon}, \beta_i^{\epsilon}) : i < m_{\epsilon}\})$$

(for $\epsilon \in \{0,1\}$) in $\mathcal{P}_\alpha$, we will say that $q^1 \leq_\alpha q^0$ iff

(*c* 1) $q^1|_1 \leq_1 q^0|_1$.

(*c* 2) $dom(F_0) \subseteq dom(F_1)$ and, for all $\xi \in dom(F_0)$ with $\xi \geq 1$, $F_1(\xi) \leq_{\mathbb{B}} F_0(\xi)$.

(*c* 3) For all $i < m_0$ there is some $\widetilde{\beta}_i \geq \beta_i^0$ such that $(N_i^0, \widetilde{\beta}_i) \in \Delta_{q^1}$.

Notation: Given $\alpha \leq \kappa$ and a $\mathcal{P}_\alpha$–condition $q = (F_q, \Delta_q)$, $dom(F_q)$ will also be denoted by $supp(q)$.

Note that if $1 \leq \alpha < \beta \leq \kappa$, then $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ and every $\mathcal{P}_\beta$–condition $q = (F, \{(N_j, \beta_j) : j < m\})$ such that $supp(q) \subseteq \alpha$ and $\beta_j \leq \alpha$ for all j is also a $\mathcal{P}_\alpha$–condition and is in fact equal to its restriction to α.

Also note that if $1 \leq \alpha \leq \kappa$ and $q \in \mathcal{P}_\alpha$, then $dom(\Delta_q) \subseteq dom(\Delta_{q|_1})$ and $q|_1 \in \mathcal{P}_1$.

Lemma

Let $\alpha \leq \beta \leq \kappa$. If $q = (F_q, \Delta_q) \in \mathcal{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathcal{P}_\beta$, and $q \leq_\alpha r|_\alpha$, then

$$r \wedge_\alpha q := (F_q \cup (F_r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\beta$ extending r. Therefore,$\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

Proof.

The crucial point of this inductive proof is the use of the markers β_i in the definition of the forcing. New side conditions (N_i, β_i) appearing in Δ_q may well have the property that $N_i \cap [\alpha, \beta) \neq \emptyset$, but they will not impose any problematic promises – coming from clause $(b4)$ in the definition – on ordinals ξ occurring in $dom(F_r \restriction [\alpha, \beta))$. The reason is simply that $\beta_i \leq \alpha$. □

Lemma

(CH) For every ordinal $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ has the $\aleph_2$–chain condition.

Definition

Given $\leq \alpha \leq \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a ctble. elem. $N \prec H(\kappa)$, we say that q is $(N, \mathcal{P}_\alpha)$–pre-generic in case

- $\alpha < \kappa$ and the pair (N, α) is in Δ_q, or else
- $\alpha = \kappa$ and the pair $(N, sup(N \cap \kappa))$ is in Δ_q.

Lemma

Suppose $1 \leq \alpha \leq \kappa$ and $N^* \in \mathcal{M}^*_\alpha$. Let $N = N^* \cap H(\kappa)$. Then the following conditions hold.

$(1)_\alpha$ *For every $q \in N$ there is some $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$–pre-generic.*

$(2)_\alpha$ *If $\mathcal{P}_\alpha \in N^*$ and $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$–pre-generic, then q is $(N^*, \mathcal{P}_\alpha)$–generic.*

Corollary

For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ is proper.

The following lemma is an adaptation of an argument due to A-C.

Lemma

Let p be an element of $\mathcal{P}_\kappa$ and $\dot{X}$ a $\mathcal{P}_\kappa$–name. Assume that p forces that $\dot{X}$ is a subset of ω_2 having order–type equal to ω_1 and $p|_1 = (c_p, \Delta_{p|_1}) \in \mathcal{P}_1$. Then, there are ordinals $\alpha_0 < \alpha_1 < \beta$ and a $\leq_\kappa$–extension t of p such that:

(1) *t forces that these three ordinals are elements of $\dot{X}$, and*

(2) *$c_t(\alpha_0, \beta) = c_t(\alpha_1, \beta)$, where $t|_1 = (c_t, \Delta_{t|_1})$.*

Corollary

For every $\alpha \leq \kappa$, $\mathcal{P}_\alpha$ *is proper.*

The following lemma is an adaptation of an argument due to A-C.

Lemma

Let p *be an element of* $\mathcal{P}_\kappa$ *and* $\dot{X}$ *a* $\mathcal{P}_\kappa$*–name. Assume that* p *forces that* $\dot{X}$ *is a subset of* ω_2 *having order–type equal to* ω_1 *and* $p|_1 = (c_p, \Delta_{p|_1}) \in \mathcal{P}_1$*. Then, there are ordinals* $\alpha_0 < \alpha_1 < \beta$ *and a* $\leq_\kappa$*–extension* t *of* p *such that:*

(1) t *forces that these three ordinals are elements of* $\dot{X}$*, and*

(2) $c_t(\alpha_0, \beta) = c_t(\alpha_1, \beta)$*, where* $t|_1 = (c_t, \Delta_{t|_1})$*.*

Proof. Wlog we may start assuming that p forces that $sup(\dot{X}) = \gamma$. Let θ be a suff. large regular cardinal and let N^* be a ctble. elem. substructure of $H(\theta)$ such that p, $\mathcal{P}_\kappa$, Φ and $\dot{X}$ are in N^*. Let $N = N^* \cap H(\kappa)$ and let p^+ be an $(N, \mathcal{P}_\alpha)$–pre-generic condition extending p. So, p^+ is $(N^*, \mathcal{P}_\alpha)$–generic. Since $cf(\gamma) = \omega_1$, $N \cap \gamma$ is bounded in γ. Find $q \leq_\kappa$–extending p^+ and an ordinal β such that $sup(N \cap \gamma) < \beta < \gamma$ and q forces that β is in $\dot{X}$.

Assume that $q|_1 = (c_q, \Delta_{q|_1})$ and let $c_r = c \restriction dom(c_q) \cap N$, $\Delta_r = \Delta_{q|_1} \cap N$. Let m be equal to the cardinality of the power set of $\Delta_{q|_1}$ and find an ordinal μ in N such that for every ordinal α in N, the pair (α, β) is not in the domain of c_q. Consider the set D of those conditions s in $\mathcal{P}_\kappa$ such that:

(a) There are $m + 1$ ordinals above δ and s forces that each of them is an element of $\dot{X}$, and

(b) $s|_1 \leq_1 (c_r, \Delta_r)$.

Proof. Wlog we may start assuming that p forces that $sup(\dot{X}) = \gamma$. Let θ be a suff. large regular cardinal and let N^* be a ctble. elem. substructure of $H(\theta)$ such that p, $\mathcal{P}_\kappa$, Φ and $\dot{X}$ are in N^*. Let $N = N^* \cap H(\kappa)$ and let p^+ be an $(N, \mathcal{P}_\alpha)$–pre-generic condition extending p. So, p^+ is $(N^*, \mathcal{P}_\alpha)$–generic. Since $cf(\gamma) = \omega_1$, $N \cap \gamma$ is bounded in γ. Find $q \leq_\kappa$–extending p^+ and an ordinal β such that $sup(N \cap \gamma) < \beta < \gamma$ and q forces that β is in $\dot{X}$.

Assume that $q|_1 = (c_q, \Delta_{q|_1})$ and let $c_r = c \restriction dom(c_q) \cap N$, $\Delta_r = \Delta_{q|_1} \cap N$. Let m be equal to the cardinality of the power set of $\Delta_{q|_1}$ and find an ordinal μ in N such that for every ordinal α in N, the pair (α, β) is not in the domain of c_q. Consider the set D of those conditions s in $\mathcal{P}_\kappa$ such that:

(a) There are $m + 1$ ordinals above δ and s forces that each of them is an element of $\dot{X}$, and

(b) $s|_1 \leq_1 (c_r, \Delta_r)$.

Since D is in N^*, D is predense below q and q is $(N^*, \mathcal{P}_\alpha)$–generic, we can find s in $D \cap N$ compatible with q. Let $\alpha_0 < \alpha_1 < \ldots < \alpha_m$ be $m+1$ ordinals (in N) witnessing that s is in D. By the pigeonhole principle, there are $i < j < m$ such that for every M in the domain of $\Delta_{q|_1}$, the pair (α_i, β) is in M iff (α_j, β). By the first antirainbow lemma of A-C, there is a condition $(c_{q'}, \Delta_{q|_1}) \leq_1$–extending $(c_q, \Delta_{q|_1})$ such that $c_{q'}(\alpha_i, \beta) = c_{q'}(\alpha_j, \beta)$. By the lemma of Properness of A-C, there is a common $\leq_1$–extension (c_t, Δ_t) of $(c_{q'}, \Delta_q)$ and $s|_1$. Finally, define $t \in \mathcal{P}_\kappa$ as the natural $\leq_\kappa$–amalgamation of (c_t, Δ_t), q and s.

Lemma

$\mathcal{P}_\kappa$ *forces* $2^{\aleph_0} = 2^{\aleph_1} = \kappa$.

Proof.

The inequalities $2^{\aleph_1} \geq \kappa$ and $2^{\aleph_0} \geq \kappa$ follow from the fact that our iteration introduces κ distinct Baumgatner clubs and κ distinct Cohen reals. The inequality $2^{\aleph_1} \leq \kappa$ follows from the fact that, by the $\aleph_2$–c.c. together with $\kappa^{\aleph_1} = \kappa$, there are exactly κ nice $\mathcal{P}_\kappa$–names for subsets of ω_1. □

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□

Lemma

$\mathcal{P}_\kappa$ *forces* MA($\mathbb{B}$).

Proof.

Note that if $\mu < \kappa$ and $(\dot{D}_i)_{i<\mu}$ is a seq. of $\mathcal{P}_\kappa$–names for dense subsets of $\mathbb{B}$, then (by the regularity of κ and the $\aleph_2$–c.c.), there is $\alpha < \kappa$ such that $(\dot{D}_i)_{i<\mu}$ appear in $V^{\mathcal{P}_\alpha}$. Finally use that $\mathcal{P}_{\alpha+1}$ is a complete suborder of $\mathcal{P}_\kappa$.

□

A topological characterization of MA($\mathbb{B}$)

It is a classical result that the covering number of the ideal of the meager sets of the Baire space is equal to the cardinal invariant $\mathfrak{m}(Cohen)$ i.e., equal to the min. cardinal κ such that there is a family $\mathcal{D}$ of dense subsets of Cohen forcing, but there is not any filter *G* on Cohen forcing meeting each of them. We will find a similar result in the context of $\mathbb{B}$. For so doing, let us start introducing a topology on the set $\mathcal{C}$ of all the normal functions from ω_1 to ω_1. Given $f \in \mathbb{B}$, define N_f as the set of all the normal functions in $\mathcal{C}$ including *f*. It is clear that the set of all N_f's is a basis for a topology on $\mathcal{C}$. From now on, we will fix this topology and we will denote by **cov**$(\mathcal{C})$ the corresponding covering number of the ideal of the meager sets (i.e., **cov**$(\mathcal{C})$ is the min. numberof nowhere dense subsets needed to cover $\mathcal{C}$).

Proposition

$\mathfrak{m}(\textit{Baumgartner}) > \omega_1$*, then* $\mathfrak{m}(\textit{Baumgartner}) = \mathbf{cov}(\mathcal{C})$*. Therefore (and under the same assumption),* $\mathrm{MA}(\mathbb{B})$ *is equivalent to the equality* $\mathbf{cov}(\mathcal{C}) = 2^{\omega_1}$*.*

Proof. First we show that $\mathfrak{m}(\textit{Baumgartner}) \leq \mathbf{cov}(\mathcal{C})$. Assume $\mathrm{MA}_\kappa(\mathbb{B})$ and let $\{X_\alpha : \alpha < \kappa\}$ be a set consisting of κ nowhere dense subsets of $\mathcal{C}$. We must show that the union of this set does not cover $\mathcal{C}$. For each α, let D_α be the set of those f in $\mathbb{B}$ that have no extensions in X_α. Since the complement of X_α contains a dense open set (in the topological sense), D_α is dense (in the partial order sense). So, $\mathrm{MA}_\kappa(\mathbb{B})$ give us a filter $G \subseteq \mathbb{B}$ meeting every D_α. Since $\mathfrak{m}(\textit{Baumgartner}) > \omega_1$, we can also assume that the union of G is an element of $\mathcal{C}$. So, this normal function is in none of the X_α.

Now we show that $\mathfrak{m}(\textit{Baumgartner}) \geq \mathbf{cov}(\mathcal{C})$. Assume that $\kappa < \mathbf{cov}(\mathcal{C})$ and let $\{D_\alpha : \alpha < \kappa\}$ be a set consisting of κ open dense subsets of $\mathbb{B}$. We must show that there is a filter G on $\mathbb{B}$ meeting each D_α. For each α, let X_α be the set of those F in $\mathcal{C}$ such that no element of D_α is included in F. Since D_α is open dense (in the partial order sense), X_α is nowhere dense (in the topological sense). Since $\kappa < \mathbf{cov}(\mathcal{C})$, we can find a normal function in $\mathcal{C} \setminus \bigcup\{X_\alpha : \alpha < \kappa\}$. It is clear that the set of all its finite approximations form a filter meeting each D_α.