GENERIC FINITE GENERATORS

Anush Tserunyan

UCLA

Introduction

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Example

For the shift action of G on k^G, letting $V_i = \{(x_g)_{g \in G} \in k^G : x_{1_G} = i\}$, $i < k$, we get that $\mathcal{P} = \{V_i\}_{i<k}$ is a k-generator.

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Another way of thinking about generators is as follows: for a Borel partition $\mathcal{P} = \{P_n\}_{n<k}$, $k \leq \infty$, define a map $f_{\mathcal{P}} : X \to k^G$ by $x \mapsto (n_g)_{g \in G}$, where $gx \in P_{n_g}$.

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$\mathcal{P}$ is a generator if and only if $f_{\mathcal{P}}$ is injective.

Fact

For $k \leq \infty$, X admits a k-generator if and only if there is a Borel G-equivariant embedding of X into k^G.

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In this talk, we are concerned with the existence of finite generators.

Overview: dynamical systems

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The goal is to find the true picture of the world (i.e. a randomly chosen $x \in X$) with probability 1. This happens precisely when $\mathcal{P}$ is a generator mod μ-NULL.

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One then defines the *time average of the entropy* of $\mathcal{P}$ by

$$h_\mu(\mathcal{P}, T) = \lim_{n\to\infty} \frac{1}{n} h_\mu\Big(\bigvee_{i<n} T^i\mathcal{P}\Big).$$

The sequence in the limit is decreasing and hence the limit is finite.

Overview: entropy and generators

Finally, the *entropy of the dynamical system* (X, μ, T) is defined as the supremum over all (finite) experiments:

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Thus, if a Polish $\mathbb{Z}$-space X admits an invariant probability measure of infinite entropy, then it does not admit a finite generator.

Is this the only obstruction to having a finite generator?

Weiss's question and a potential dichotomy

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I show that **these questions are actually equivalent**, so a positive answer to Weiss's question would imply the following

Potential dichotomy

For any aperiodic Polish $\mathbb{Z}$-space X,

either *there is an invariant probability measure of infinite entropy,*

or else *there is a finite generator.*

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Theorem (Ts.)

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Measure theoretic context: the Krengel-Kuntz theorem

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Theorem (Krengel, Kuntz, '74)

Let X be a Polish G-space and let μ be a quasi-invariant Borel probability measure on X (i.e. G preserves the μ-null sets). If there is no invariant probability measure, then X admits a 2-generator modulo μ-NULL.

Baire category context: Kechris's question

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The nonexistence of invariant measures is not mentioned in the hypothesis of the question because it is automatic, due to the following:

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Thus, a positive answer to Weiss's question (for the general case) would imply a positive answer to Kechris's question.

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For the rest of the talk, we will discuss the proof of this theorem.

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- Show that for a generic α, $\mathcal{P}_\alpha$ works, i.e. $\forall^* \alpha \forall^* x \Phi(\mathcal{P}_\alpha, x)$.
- By the Kuratowski-Ulam theorem, it is enough to show that for a generic $x \in X$ a generic α works: $\forall^* x \forall^* \alpha \Phi(\mathcal{P}_\alpha, x)$.

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The latter is often an easier task since it allows one to work locally with a fixed $x \in X$.

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Transform this 2-dimensional problem into two 1-dimensional problems.

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Example

For the translation action of $\mathbb{Z}$ on $\mathbb{R}$, the set $[0, 2]$ is weakly wandering.

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A Borel set $W \subseteq X$ is called *weakly wandering* if there is $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ such that $g_n W$ are pairwise disjoint.

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However, using the marker lemma, we get:

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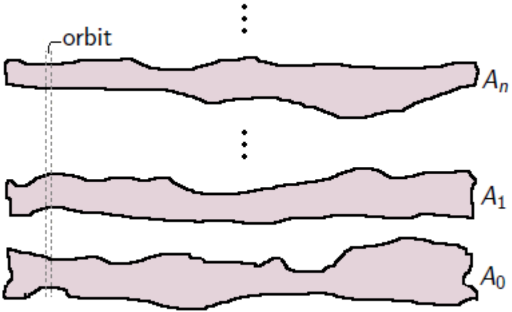
For any aperiodic action $G \curvearrowright X$, there is a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint Borel complete sections. We call it a disjoint marker sequence.



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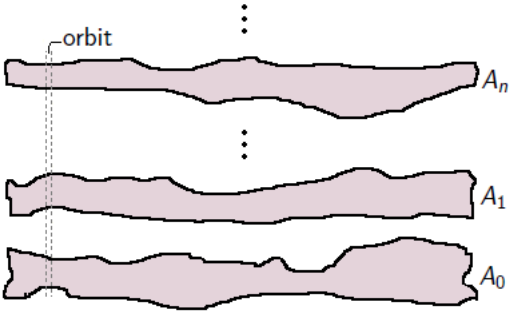

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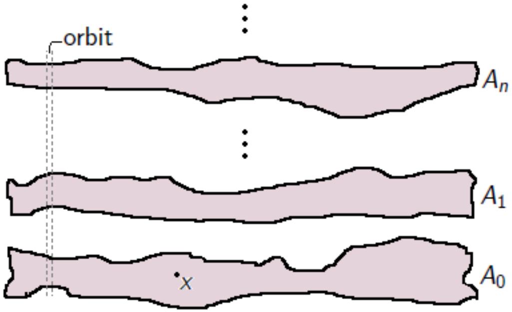

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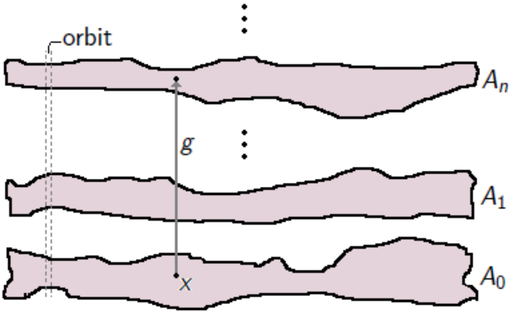

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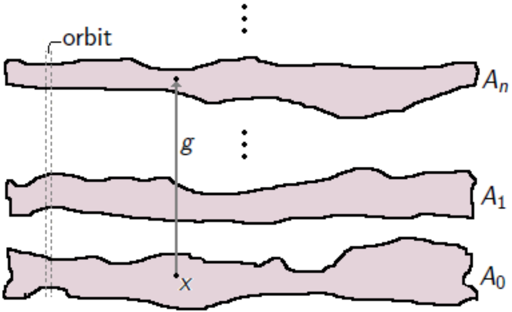

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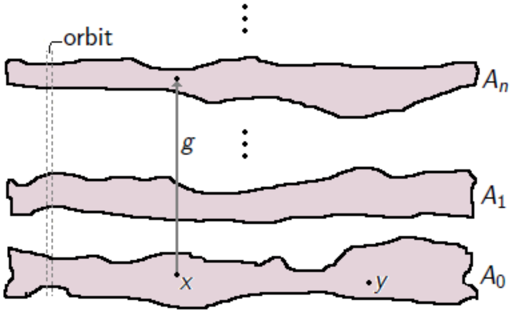

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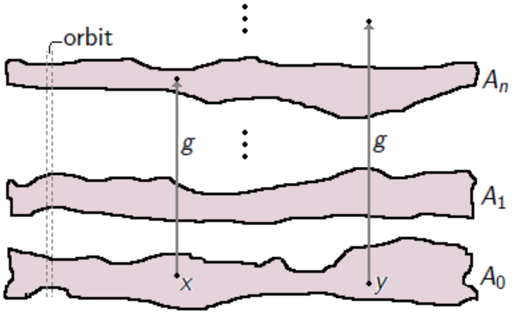

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First 1-dimensional problem

Recall: For a family $\mathcal{F}$ of subsets of X, $\sigma_G(\mathcal{F})$ denotes the σ-algebra generated by the set $G\mathcal{F} = \{gA : g \in G, A \in \mathcal{F}\}$ of G-translates of $\mathcal{F}$. If $\mathcal{F} = \{A\}$, write $\sigma_G(A)$.

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Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint marker sequence and fix an open basis $(U_n)_{n\in\mathbb{N}}$. Then, for a generic $\alpha \in \mathbb{N}^{\mathbb{N}}$, the set

$$A = \bigcup_{n\in\mathbb{N}} A_n \cap U_{\alpha(n)}$$

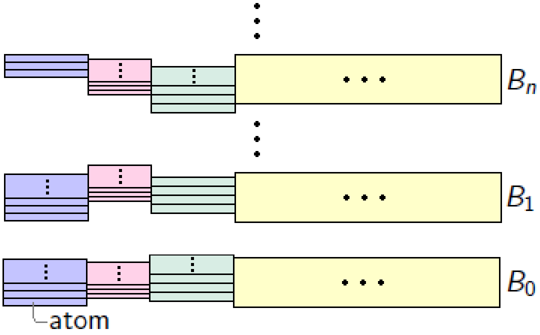
works. □

$\sigma_G(A)$-measurable disjoint marker sequence

From that lemma, we get the following:

Proposition

Modulo a meager set, there is a disjoint sequence $(B_n)_{n\in\mathbb{N}}$ of Borel complete sections with $B_n \in \sigma_G(A)$.



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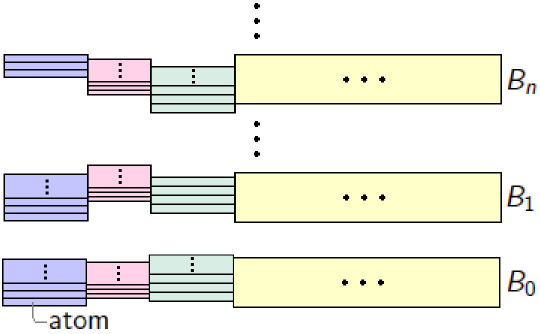

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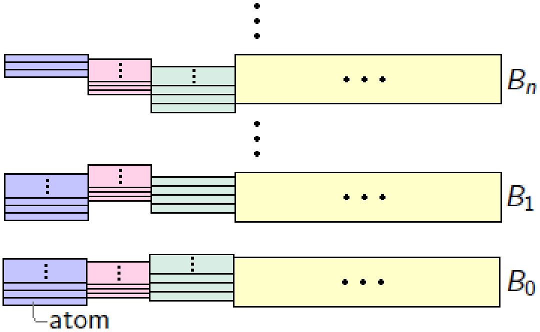

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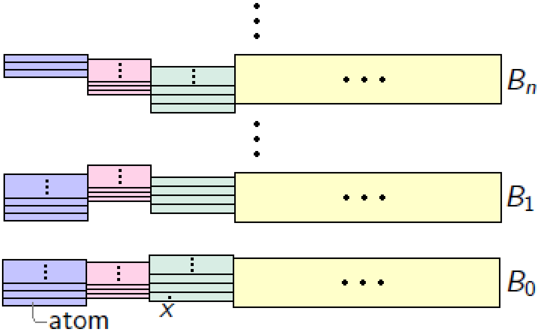

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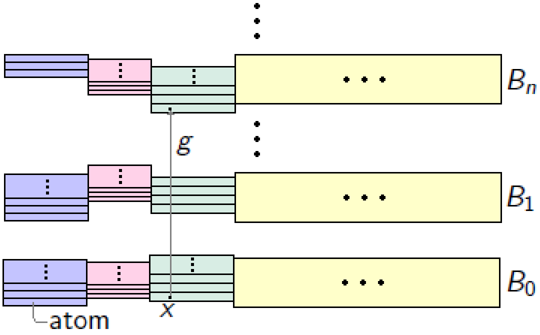

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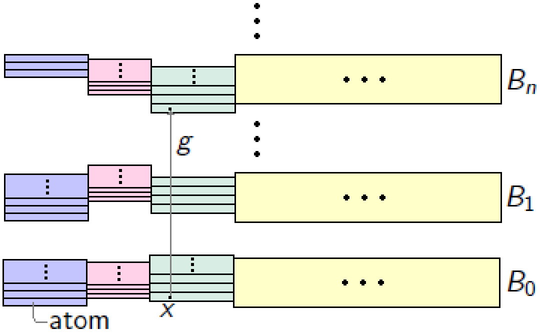

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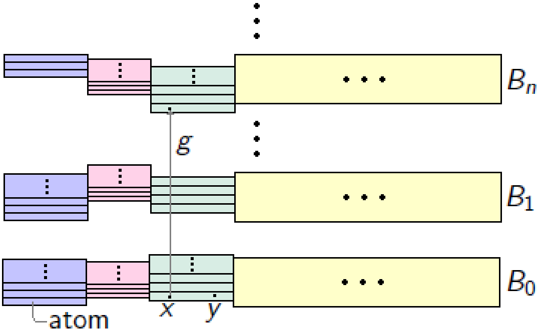

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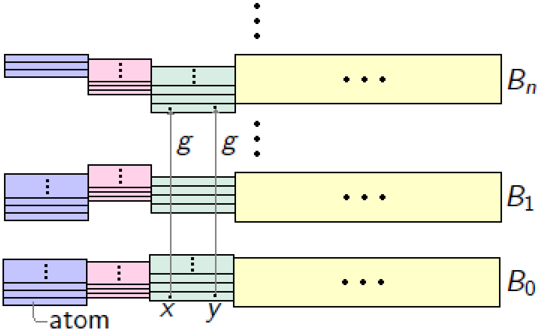

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As expected, the partition $\mathcal{P}$ generated by $\{A, B\}$ is a generator (modulo a meager set) and contains at most 4 elements, so we are done.

THANK YOU

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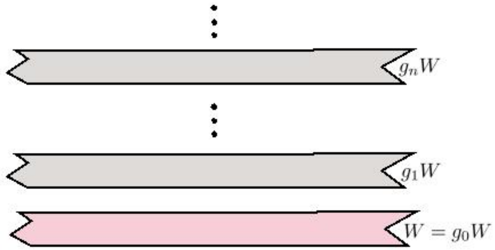

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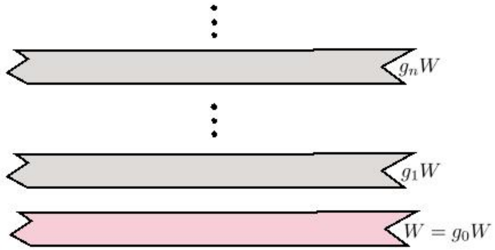

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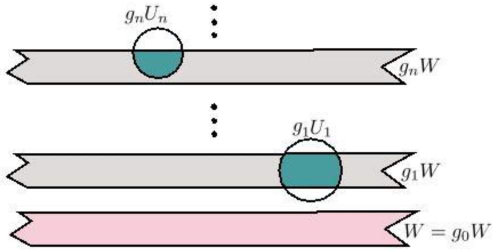

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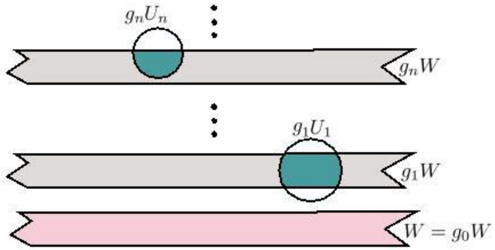

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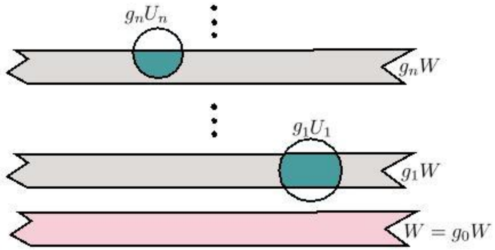

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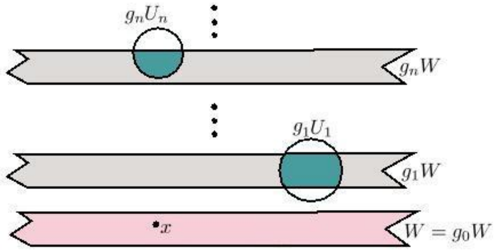

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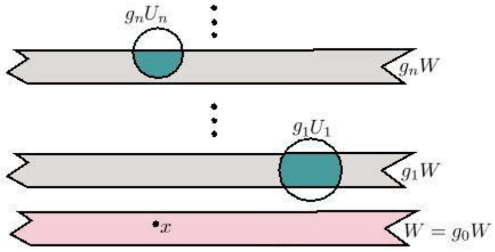

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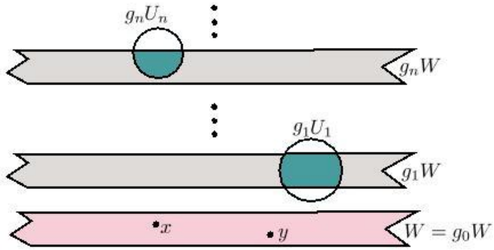

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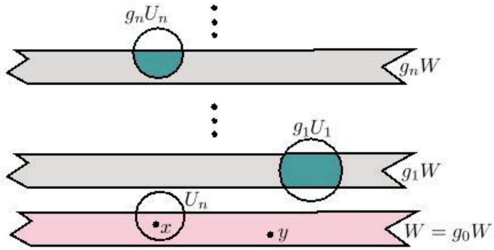

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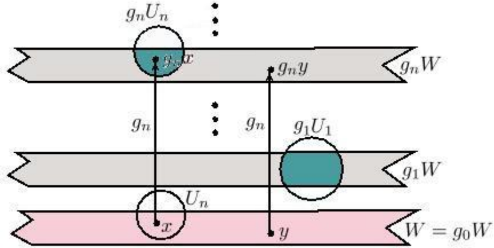

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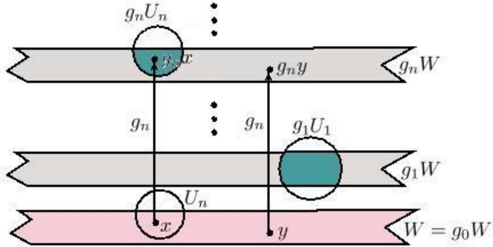

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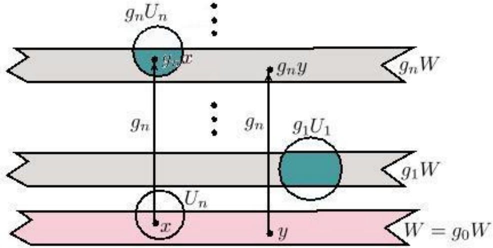

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