Generic absoluteness, strong forcing axioms, MM^{+++}

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Waterloo
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2. Why forcing axioms are so effective in settling most of the problems of set theory or of mathematics which are undecidable on the basis of ZFC alone?

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- Truth in $M^{\mathbb{B}}$ is (almost) definable in M with parameter $\mathbb{B}$.

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What is difficult is to define $[\![\tau_1 \in \tau_2]\!]_{\mathbb{B}}$ and $[\![\tau_1 = \tau_2]\!]_{\mathbb{B}}$.

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Theorem (Cohen)

Assume M is a (transitive) model of ZFC *and* $\mathbb{B} \in M$ *is a non atomic complete boolean algebra in M. Then*

$$M \models (\llbracket \phi \rrbracket_{\mathbb{B}} = 1_{\mathbb{B}})$$

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What else holds in $M^{\mathbb{B}}$ depends on the choice of $\mathbb{B}$ and the first order properties of M.

Baire's category theorem and forcing

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$G \subset \mathbb{B}^+$ is a ultrafilter on $\mathbb{B}$ if:

- for all $p \in G$ and $q \geq_\mathbb{B} p$, $q \in G$,
- $1_\mathbb{B} \in G$,
- for all $p, q \in G$, $p \wedge_\mathbb{B} q \in G$,
- for all $p \in \mathbb{B}^+$, $p \in G$ or $\neg_\mathbb{B} p \in G$.

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$A \subset \mathbb{B}^+$ is open in $\mathbb{B}$ if whenever $p \in A$ and $q \leq p$, $q \in A$ as well.

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Theorem (Baire's category theorem)

For all non atomic complete boolean algebras $\mathbb{B}$, $\mathsf{FA}_{\aleph_0}(\mathbb{B})$ holds.

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Theorem (Los Theorem for boolean valued models)

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Then we can define the quotient structure $M^{\mathbb{B}}/G$ *letting*

$$[\tau]_G R_G [\sigma]_G$$

if and only if $[\![\tau R \sigma]\!]_{\mathbb{B}} \in G$ *and we get that*

$$(M^{\mathbb{B}}/G, E_G) \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$$

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Baire's category theorem is irrelevant for the construction of a Tarski model of ZFC $+ \phi$......

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*Then we also have that the transitive collapse of $M^{\mathbb{B}}/G$ is the **transitive structure** $M[G]$ and with this identification the evaluation map*

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- *$\sigma_G(\check{a}) = a = [\check{a}]_G$ for all $a \in M$.*
- *If $N \models$ ZFC, $G \in N$, and $M \subset N$, then $M[G] \subset N$,*

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Forcing in its usual development focuses on partial orders and not on their boolean completions, however:

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Any partial order P gives rise in a natural way to its boolean completion $\mathbb{B}(P)$ which is a non-atomic complete boolean algebra if P has no minimal elements (i.e. P is a non trivial notion of forcing).
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For M a transitive model of ZFC and P and Q in M non trivial forcing notions TFAE:

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Non-atomic complete boolean algebras are sufficient to capture the class of models produced by forcing.

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Forcing is a powerful tool to prove theorems and transforms, for certain class of problems, a proof of the consistency of a solution, in the solution.

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(We needed the upward absoluteness of $\phi(a, r)$ to conclude the proof of Cohen's absoluteness.)

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Woodin's absoluteness results show that large cardinal axioms give a correct and complete semantics with respect to first order derivability and forceability for the theory of $H_{\aleph_1} \subset L(\mathrm{Ord}^{\omega})$ with real parameters.
A key ingredient of Woodin's result is the fact that $\mathrm{FA}_{\aleph_0}(\mathbb{B})$ holds for the largest possible class of boolean algebras $\mathbb{B}$, i.e. for all $\mathbb{B}$.

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Let V be a model of ZFC. The Σ_0-diagram of $H^V_{\omega_2}$ is given by the theory

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$\Omega_{\aleph_1}$ and absoluteness

Lemma (Generalized Cohen's absoluteness Lemma)

Assume $\mathsf{FA}_{\aleph_1}(\mathbb{B})$ *and* $\phi(x, y)$ *is a* Σ_0*-formula. The following are equivalent for some* $a \in H_{\omega_2}$*:*

1. $H_{\omega_2} \models \exists x \phi(x, a)$,
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KEY OBSERVATION: $\Omega^V_{\aleph_1}$ gives a complete and correct semantics for the Σ_1-theory of $H^V_{\aleph_2}$.

Stationary set preserving forcings and absoluteness

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Assume that $M \supset V$ is a model of ZFC *which maintains the truth of the Σ_0-diagram of V. Then $\mathrm{NS}_{\omega_1}^M \cap V = \mathrm{NS}_{\omega_1}$.*

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Fact (Shelah)

Assume $\mathbb{B} \notin \mathrm{SSP}$ and G is V-generic for $\mathbb{B}$. Then $V[G]$ falsifies the Σ_0-elementary diagram of $H_{\omega_2}^V$.

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$$[\![\check{S} \text{ is stationary}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Fact (Shelah)

Assume $\mathbb{B} \notin \mathrm{SSP}$ *and* G *is* V*-generic for* $\mathbb{B}$*. Then* $V[G]$ *falsifies the* Σ_0*-elementary diagram of* $H^V_{\omega_2}$.

Corollary

$\Omega_{\aleph_1} \subseteq \mathrm{SSP}$

Enlarging $\Omega_{\aleph_1}$ to become SSP.

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Following these patterns of ideas strong forcing axioms have been discovered and proved consistent.

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Baire's category theorem (i.e. $\mathsf{FA}_{\aleph_0}(\mathbb{B})$ holds for all $\mathbb{B}$) has been a key ingredient in Woodin's absoluteness result for $L(\mathrm{Ord}^{\omega})$.

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MM is the natural extension to ω_1 of Baire's category theorem.
This makes plausible that Woodin's absoluteness result can be stepped up to $L(\mathrm{Ord}^{\omega_1})$ for some theory extending

$$\mathsf{ZFC} + \mathsf{MM} + \text{ large cardinals.}$$

In what follows we shall show that this is indeed the case.

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We shall present MM^{+++}, a natural strengthening of MM, and show that it makes the theory of $L(\mathrm{Ord}^{\omega_1})$ provably complete with respect to SSP-forcings and first order calculus.

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On the other hand, Aspero, Larson and Moore have shown that no such generic absoluteness result can be produced for any theory extending $\mathsf{ZFC} + \mathsf{CH}$.

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- Generic extensions by stationary set preserving forcings corresponds to complete boolean algebras which are stationary set preserving.
- Iterations of stationary set preserving forcings correspond to a natural family of directed systems of complete homomorphisms.

Definition

$\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ is the category whose objects are SSP complete boolean algebras $\mathbb{B}$ and whose arrows are non-atomic complete (*but possibly non-injective*) homomorphisms

$$i : \mathbb{B} \to \mathbb{Q}$$

such that

$$[\![\mathbb{Q}/i[\dot{G}_{\mathbb{B}}] \in \mathsf{SSP}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

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Fact

If Γ is the class of all complete non-atomic boolean algebras and Θ is the class of all non-atomic complete homomorphisms between elements of Γ we have that any two elements $P, Q \in \Gamma$ are compatible in $\mathbb{U}^{\Gamma,\Theta}$ as witnessed by $Col(\omega, < \delta)$ for any large enough $\delta > |P|, |Q|$.

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Fact

If $P = Col(\omega_1, \omega_2)$ and Q is Namba forcing on $\aleph_2$, $\mathbb{B}(P)$ and $\mathbb{B}(Q)$ are incompatible conditions of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

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Proof: If not in some generic extension of an SSP forcing which absorbs both of them we would have that ω_2^V has at the same time countable and uncountable cofinality.

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If $\mathbb{U}_\delta \in \mathsf{SSP}$, $\mathbb{U}_\delta$ is totally rigid.

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Assume there are class many supercompact cardinals. Then the class

$$D_0 = \{\mathbb{B} : \mathbb{B} \textit{ is totally rigid}\}$$

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MM^{+++} asserts that $D_0 \cap D_1$ is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$.

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Fact

Let $\mathbb{B} \in \mathrm{SSP}$ *be a presaturated tower and* G *be* V*-generic for* $\mathbb{B}$*. Then*

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V \rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^V \rangle$$

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Assume

$$T \supseteq \mathrm{ZFC} + \mathrm{MM}^{+++} \textit{ there are class many superhuge cardinals,}$$

ϕ is any formula and $a \subset \omega_1$. Then the following are equivalent:

1. $T \vdash \phi^{L(\mathrm{Ord}^{\omega_1})}(a)$
2. *T proves that there is some $\mathbb{B} \in \mathrm{SSP}$ such that*

$$[\![\phi^{L(\mathrm{Ord}^{\omega_1})}(a)]\!]_{\mathbb{B}} = [\![\mathrm{MM}^{+++}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Similarity properties of $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$

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Observe that if $P \in \mathsf{SSP}$ and G is V-generic for P

$$(\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}})^{V[G]} = \{\mathbb{B}/i[G] : V \models i : P \to \mathbb{B} \text{ is a complete embedding }\}$$

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Proposition

Let $\mathbb{U}_\delta = \mathbb{U}^{\text{SSP,SSP}} \cap V_\delta$ and $\mathbb{B} \in \mathbb{U}_\delta$. Assume $i : \mathbb{B} \to \mathbb{B}(\mathbb{U}_\delta)$ is a complete homomorphism with a stationary set preserving quotient. Then

$$[\![\dot{\mathbb{U}}_\delta = (\mathbb{U}_\delta \restriction \mathbb{B})/i[\dot{G}_\mathbb{B}]]\!]_\mathbb{B} = 1_\mathbb{B}.$$

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$$[\![\dot{\mathbb{U}}_\delta = (\mathbb{U}_\delta \restriction \mathbb{B})/i[\dot{G}_\mathbb{B}]]\!]_\mathbb{B} = 1_\mathbb{B}.$$

This is the same similarity property that is peculiar of $Col(\omega, < \delta)$.

How to get there? *I*

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δ is superhuge if for all $\lambda > \delta$ there is $j : V \to M$ with $M^{j(\delta)} \subset M \subset V$ and $j(\delta) > \lambda$.

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MM^{+++} *is consistent relative to the existence of class many superhuge cardinals.*

Actually any of the known iteration of length a super-huge δ which produces a model of MM^{++}, will produce a model of MM^{+++}.

How to get there? *II*

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Theorem (V.)

Assume MM^{+++} *and that there are class many superhuge cardinals* δ*.*
Then for any such δ*:*

- $\mathbb{U}_\delta \in$ SSP *is a totally rigid presaturated tower (i.e.* $\mathbb{U}_\delta \in D_0 \cap D_1$*).*

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- $\mathbb{U}_\delta \in$ SSP *is a totally rigid presaturated tower (i.e.* $\mathbb{U}_\delta \in D_0 \cap D_1$*).*
- $\mathbb{B} \geq_{\text{SSP}} \mathbb{U}_\delta \restriction \mathbb{B}$ *for all* $\mathbb{B} \in \mathbb{U}_\delta$*.*

A sketch of proof of the generic absoluteness result

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Then:

- $\mathbb{U}_\delta^{V[H]} = (\mathbb{U}_\delta \restriction P)^V / i[H]$

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Let $i : P \to \mathbb{U}_\delta \restriction P$ be a complete embedding and $H = i^{-1}[G] \in V[G]$ be V-generic for P.

Then:

- $\mathbb{U}_\delta^{V[H]} = (\mathbb{U}_\delta \restriction P)^V / i[H]$
- δ is superhuge in V, $V[H]$ which are both models of MM^{+++}.

Thus $\mathbb{U}_\delta \restriction P$ is a presaturated tower in V and $\mathbb{U}_\delta^{V[H]}$ is a presaturated tower in $V[H]$.

This gives that:

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V\rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^V\rangle.$$

$$\langle L(\mathrm{Ord}^{\omega_1})^{V[H]}, \in, P(\omega_1)^{V[H]}\rangle \prec \langle L(\mathrm{Ord}^{\omega_1})^{V[G]}, \in, P(\omega_1)^{V[H]}\rangle.$$

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Thus

$$\langle L(\mathrm{Ord}^{\omega_1})^V, \in, P(\omega_1)^V \equiv \langle L(\mathrm{Ord}^{\omega_1})^{V[H]}, \in, P(\omega_1)^V\rangle.$$

Ideas for the proof of the consistency of MM^{+++}

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Theorem

Assume δ is super huge and P_δ is any of the standard methods to produce a model of MM^{++} collapsing δ to become ω_2. Let G be V-generic for P_δ. Then $V[G]$ models that $j(P_\delta)/G$ is totally rigid and forcing equivalent to a presaturated normal tower.

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Corollary

Assume δ is super huge and P_δ is any of the standard methods to produce a model of MM^{++} collapsing δ to become ω_2. Let G be V-generic for P_δ. Then in $V[G]$ the class $\mathsf{SPT} = D_0 \cap D_1$ of totally rigid presaturated normal towers is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$, i.e. MM^{+++} holds.

Ideas for the proof that $\mathbb{U}_\delta$ is a strongly presaturated tower

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Theorem

Assume SPT *is a dense class in* $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$. *Let* δ *be an inaccessible cardinal such that* $\mathbb{U}_\delta \in \mathsf{SSP}$ *and* $\mathsf{SPT} \cap V_\delta$ *is dense in* $\mathbb{U}_\delta$. *Then* $\mathbb{U}_\delta$ *is forcing equivalent to a presaturated normal tower.*

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Theorem

Assume SPT *is a dense class in* $\mathbb{U}^{\text{SSP,SSP}}$. *Let* δ *be an inaccessible cardinal such that* $\mathbb{U}_\delta \in$ SSP *and* SPT $\cap V_\delta$ *is dense in* $\mathbb{U}_\delta$. *Then* $\mathbb{U}_\delta$ *is forcing equivalent to a presaturated normal tower.*

Sketch of proof: We use the density of SPT in $\mathbb{U}_\delta$ to show that whenever G is V-generic for $\mathbb{U}_\delta$, we can patch together the generic filters for elements of $G \cap$ SPT to define in $V[G]$ a generic ultrapower embedding

$$j : V \to M$$

such that $M^{<\delta} \subset M$, $crit(j) = \omega_2$, $j(\omega_2) = \delta$.
This will give that $\mathbb{U}_\delta$ is forcing equivalent to a presaturated tower of normal filters.

Now we observe that if SPT is dense in $\mathbb{U}^{\mathsf{SSP},\mathsf{SSP}}$ and δ is a strong cardinal which is a limit of $< \delta$-supercompact cardinal we have that $\mathbb{U}_\delta \in$ SSP is totally rigid and forcing equivalent to a presaturated tower of normal ideals.

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This concludes the sketch of all proofs.

SOME REFERENCES

In these two papers are presented the results I talked about:

- **Martin's maximum revisited**
- **Category forcings,** MM^{+++}**, and generic absoluteness for the theory of strong forcing axioms**

They're both available on my webpage:
`http://www2.dm.unito.it/paginepersonali/viale/`.

THANKS FOR YOUR PATIENCE AND ATTENTION!!