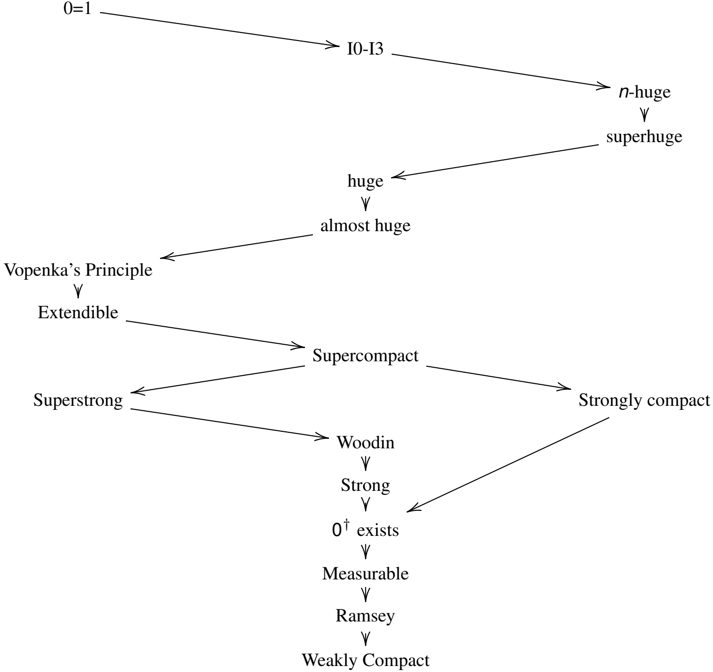
Strong Combinatorial Properties at Small Cardinals

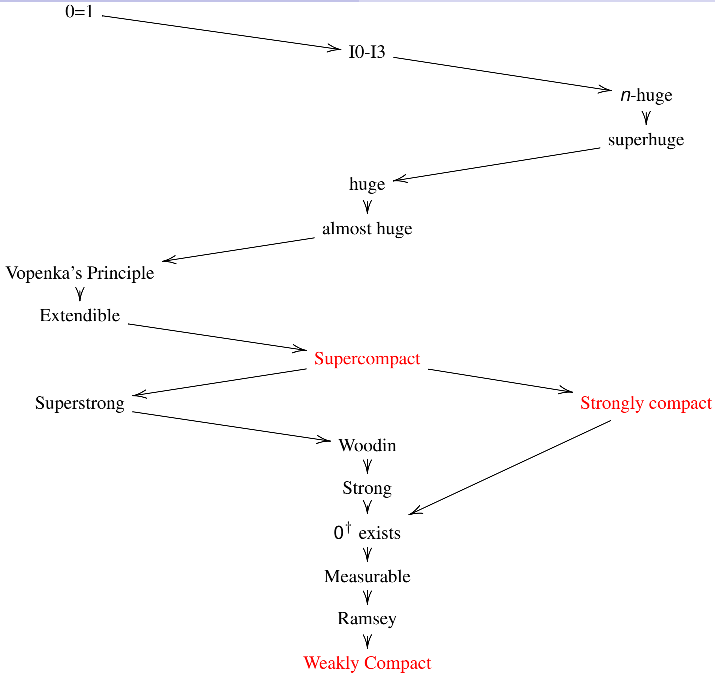
2013 North American Annual Meeting

Laura Fontanella

Kurt Gödel Research Center for Mathematical Logic
University of Vienna
http://www.logique.jussieu.fr/∼fontanella

08/05/13


0=1
I0-I3
n-huge
superhuge
huge
almost huge
Vopenka's Principle
Extendible
Supercompact
Superstrong
Strongly compact
Woodin
Strong
$0^\dagger$ exists
Measurable
Ramsey
Weakly Compact


0=1
I0-I3
n-huge
superhuge
huge
almost huge
Vopenka's Principle
Extendible
Supercompact
Superstrong
Strongly compact
Woodin
Strong
0† exists
Measurable
Ramsey
Weakly Compact

Erdös & Tarski 1961

κ is weakly compact iff it is inaccessible and it satisfies the tree property.

Jech 1973, Di Prisco & Zwicker 1980, Donder & Weiss 2010

κ is strongly compact iff it is inaccessible and it satisfies the strong tree property.

Jech 1973, Magidor 1974, Donder & Weiss 2010

κ is supercompact iff it is inaccessible and it satisfies the super tree property.

Erdös & Tarski 1961

κ is weakly compact iff it is inaccessible and it satisfies the tree property.

Jech 1973, Di Prisco & Zwicker 1980, Donder & Weiss 2010

κ is strongly compact iff it is inaccessible and it satisfies the strong tree property.

Jech 1973, Magidor 1974, Donder & Weiss 2010

κ is supercompact iff it is inaccessible and it satisfies the super tree property.

Erdös & Tarski 1961

κ is weakly compact iff it is inaccessible and it satisfies the tree property.

Jech 1973, Di Prisco & Zwicker 1980, Donder & Weiss 2010

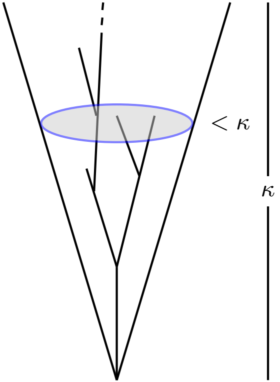
κ is strongly compact iff it is inaccessible and it satisfies the strong tree property.

Jech 1973, Magidor 1974, Donder & Weiss 2010

κ is supercompact iff it is inaccessible and it satisfies the super tree property.

The Tree Property

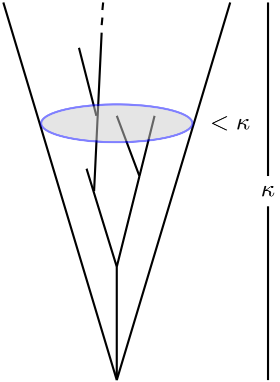
A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.



The Tree Property

A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.

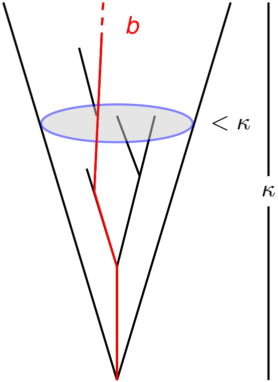
A regular cardinal κ satisfies the tree property if, and only if, every κ-tree has a cofinal branch.



The Tree Property

A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.

A regular cardinal κ satisfies the tree property if, and only if, every κ-tree has a cofinal branch.



The Tree Property

Definition

A κ-Aronszajn tree is a κ-tree with no cofinal branches.

Theorem

- (König's Lemma 1936) $\aleph_0$ satisfies the tree property;
- (Aronszajn 1934) $\aleph_1$ does not satisfy the tree property;
- (Specker 1949) If $\tau^{<\tau} = \tau$, then the tree property fails at τ^+;
- (Mitchell 1972) If *Cons*(*ZFC* $+ \exists\kappa$ weakly compact), then for every regular τ such that $\tau^{<\tau} = \tau$, we have *Cons*(*ZFC* $+ \tau^{++}$ has the tree property).

The Tree Property at *Small Cardinals*

Mitchell 1972

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ weakly compact})$, then $Cons(ZFC + \aleph_n \text{ has the Tree Property })$.

Cummings & Foreman 1998

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \forall n \geq 2\ (\aleph_n \text{ has the tree property}))$.

Magidor & Shelah 1996, Sinapova 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \aleph_{\omega+1} \text{ has the tree property})$.

The Tree Property at Small Cardinals

Neeman 2012

If *Cons*(*ZFC* $+ \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then
Cons(*ZFC* $+$ every regular cardinal $\leq \aleph_{\omega+1}$ has the tree property).

Friedman & Halilović 2011

If *Cons*(*ZFC* $+ \exists \kappa$ weakly compact hypermeasurable), then
Cons(*ZFC* $+ \aleph_{\omega+2}$ has the tree property).

Open question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the tree property?

The tree property at the successor and the double successor of a singular cardinal

Unger 2013

Cons(*ZFC* $+ \exists\kappa$ supercompact $+ \exists\mu > \kappa$ weakly compact) implies *Cons*(*ZFC* $+$ κ is singular strong limit of cofinality ω, SCH fails at κ, there are no special κ^+-Aronszajn trees and κ^{++} has the tree property).

Fontanella & Friedman 2013

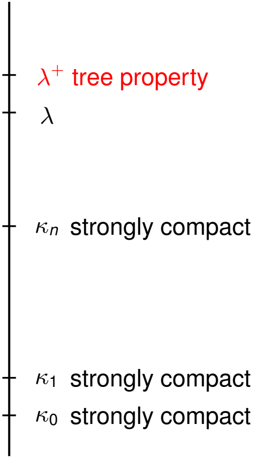
Cons(*ZFC* $+ \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals $+ \exists\mu$ weakly compact above $\lambda := \sup_{n<\omega}\kappa_n$) implies *Cons*(*ZFC* $+$ λ^+ and λ^{++} have the tree property and $2^{\kappa_0} = 2^{\lambda} = \lambda^{++}$).

Fontanella & Friedman 2013

Cons(*ZFC* $+ \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals $+ \exists\mu$ weakly compact above $\sup_{n<\omega}\kappa_n$) implies *Cons*(*ZFC* $+$ $\aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property).

Magidor & Shelah 1996

Let $\langle \kappa_n \rangle_{n<\omega}$ be a sequence of strongly compact cardinals and let $\lambda := \sup_{n<\omega} \kappa_n$, then λ^+ has the tree property.



Fontanella & Friedman 2013

Cons(*ZFC* $+ \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\lambda := \sup_{n<\omega} \kappa_n$) implies *Cons*(*ZFC* $+$ λ^+ and λ^{++} have the tree property and $2^{\kappa_0} = 2^\lambda = \lambda^{++}$).

μ weakly compact

λ^+ tree property

$\lambda \ = \sup_{n<\omega} \kappa_n$

κ_n (indestructible) supercompact

κ_1 (indestructible) supercompact

κ_0 (indestructible) supercompact

Fontanella & Friedman 2013

$Cons(ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\lambda := \sup_{n<\omega} \kappa_n)$ implies $Cons(ZFC +$ λ^+ and λ^{++} have the tree property and $2^{\kappa_0} = 2^\lambda = \lambda^{++})$.

μ inaccessible + tree property

λ^+ tree property

$\lambda = \sup_{n<\omega} \kappa_n$

κ_n (indestructible) supercompact

κ_1 (indestructible) supercompact

κ_0 (indestructible) supercompact

Fontanella & Friedman 2013

Cons(*ZFC* $+\exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals $+\exists\mu$ weakly compact above $\lambda := \sup_{n<\omega}\kappa_n$) implies *Cons*(*ZFC* $+$ λ^+ and λ^{++} have the tree property and $2^{\kappa_0} = 2^\lambda = \lambda^{++}$).

$\mu = \lambda^{++}$ + tree property

λ^+ tree property

$\lambda = \sup_{n<\omega}\kappa_n$

$\mathbb{M}$ collapses all cardinals between λ^+ and μ; it makes $2^{\kappa_0} \geq \mu$ (hence $2^\lambda \geq \mu$)

κ_n old supercompact

κ_1 old supercompact

κ_0 old supercompact

Mitchell's forcing over a singular cardinal

Definition

Conditions of $\mathbb{M}$ are pairs (p, q) such that

1. $p \in Add(\kappa_0, \mu)$;
2. q is a function of size $\leq \lambda$ such that every $\alpha \in dom(q)$ is a cardinal between λ^+ and μ, and $\Vdash_{Add(\kappa_0,\alpha)} q(\alpha) \in Add(\lambda^+, 1)$.

We let $(p, q) \leq (p', q')$ if and only if $p \leq p'$, $dom(q') \subseteq dom(q)$ and for every $\alpha \in dom(q')$ $p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

Fact

$\mathbb{M}$ is a projection of $Add(\kappa_0, \mu) \times \mathbb{Q}$ where $\mathbb{Q} := \{(0, q);\ (0, q) \in \mathbb{M}\}$ is λ^+-directed closed.

Mitchell's forcing over a singular cardinal

Definition

Conditions of $\mathbb{M}$ are pairs (p, q) such that

1. $p \in Add(\kappa_0, \mu)$;
2. q is a function of size $\leq \lambda$ such that every $\alpha \in dom(q)$ is a cardinal between λ^+ and μ, and $\Vdash_{Add(\kappa_0,\alpha)} q(\alpha) \in Add(\lambda^+, 1)$.

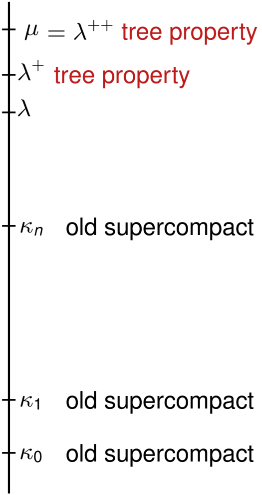
We let $(p, q) \leq (p', q')$ if and only if $p \leq p'$, $dom(q') \subseteq dom(q)$ and for every $\alpha \in dom(q')$ $p \restriction \alpha \Vdash q(\alpha) \leq q'(\alpha)$.

Fact

$\mathbb{M}$ is a projection of $Add(\kappa_0, \mu) \times \mathbb{Q}$ where $\mathbb{Q} := \{(0, q);\ (0, q) \in \mathbb{M}\}$ is λ^+-directed closed.

Fontanella & Friedman 2013

Cons(*ZFC* $+ \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\sup_{n<\omega} \kappa_n$) implies *Cons*(*ZFC* $+ \aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property).



Fontanella & Friedman 2013

$Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals $+ \exists\mu$ weakly compact above $\sup_{n<\omega}\kappa_n)$ implies $Cons(ZFC + \aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property$)$.

$\mu = \lambda^{++} = \aleph_{\omega+2}$ tree property

$\lambda^{+} = \aleph_{\omega+1}$ tree property

$\lambda = \aleph_\omega$

$\kappa_n = \aleph_{n+2}$

$\kappa_1 = \aleph_3$

$\kappa_0 = \aleph_2$

Fontanella & Friedman 2013

Cons($ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals $+ \exists \mu$ weakly compact above $\sup_{n<\omega} \kappa_n$) implies *Cons*($ZFC + \aleph_{\omega+1}$ has the tree property and $\aleph_{\omega+2}$ has the tree property).

$\mu = \lambda^{++} = \aleph_{\omega+2}$ tree property

$\lambda^{+} = \aleph_{\omega+1}$ tree property

$\lambda = \aleph_\omega$

$\kappa_n = \aleph_{n+2}$

$\kappa_1 = \aleph_3$

$\kappa_0 = \aleph_2$

There exists $\nu < \kappa_0$ singular strong limit of cofinality ω, such that

$$Coll(\omega, \nu) \times Coll(\nu^+, < \kappa_0)$$

$$\times \prod_{n<\omega} Coll(\kappa_n, < \kappa_{n+1}) \times \mathbb{M}$$

forces the tree property at $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$.

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

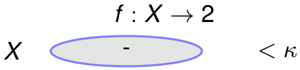
1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

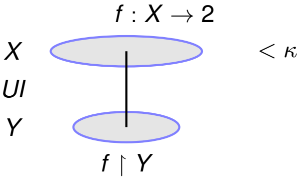


The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

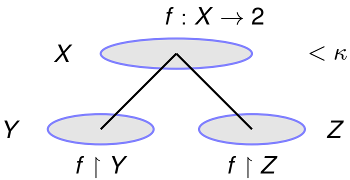


The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.



The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \restriction X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \restriction X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

The Super Tree Property

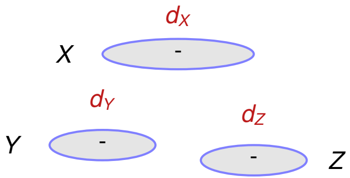
Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.



The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An ineffable branch for D is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \restriction X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An ineffable branch for D is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \restriction X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa}\rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa}\rangle$ an F-level sequence. An ineffable branch for D is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

Open Question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the strong or the super tree properties?

Weiss 2010

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then
$Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

Fontanella 2012, Unger 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then
$Cons(ZFC + \forall n \geq 2,\ \aleph_n \text{ has the Super Tree Property })$.

Fontanella 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then
$Cons(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property })$.

Open Question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the strong or the super tree properties?

Weiss 2010

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

Fontanella 2012, Unger 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n \text{ has the Super Tree Property })$.

Fontanella 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property })$.

The Strong Tree Property at $\aleph_{\omega+1}$

Fontanella 2012

If *Cons*($ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then *Cons*($ZFC + \aleph_{\omega+1}$ has the Strong Tree Property).

The Strong Tree Property at $\aleph_{\omega+1}$

Magidor & Shelah 1996

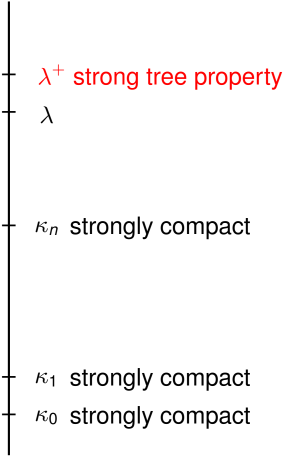
If λ is a singular limit of strongly compact cardinals, then λ^+ satisfies the Tree Property.

Fontanella 2012 - Key Lemma

If λ is a singular limit of strongly compact cardinals, then λ^+ satisfies the Strong Tree Property.

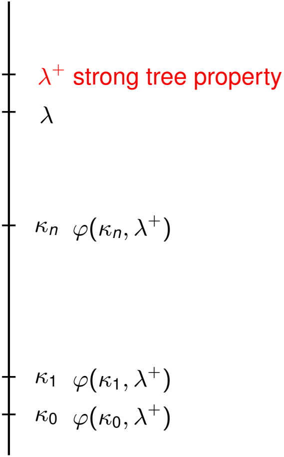
Fontanella - Key Lemma

If λ is a singular limit of strongly compact cardinals, then λ^+ satisfies the Strong Tree Property.



Fontanella - Key Lemma

If λ is a singular limit of strongly compact cardinals, then λ^+ satisfies the Strong Tree Property.



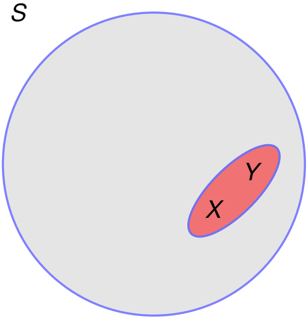
A Partition Property for Strongly Compact Cardinals

For a cofinal $S \subseteq [\mu]^{<\nu}$, we denote by $[[\ S\ ]]^2$ the set of all pairs $(X, Y) \in S \times S$ such that $X \subseteq Y$.

Definition

Let $\lambda \geq \kappa$, the principle $\varphi(\kappa, \lambda)$ establishes that for every $\mu \geq \lambda$ and for every stationary $S \subseteq [\mu]^{<\lambda}$, every $c : [[\ S\ ]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H of color $i < \gamma$ which is also stationary, i.e.

$$\text{for every } X, Y \in H \text{ there is } Z \supseteq X, Y \text{ in } H \text{ such that } c(X, Z) = i = c(Y, Z).$$



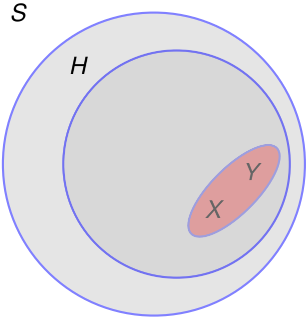
A Partition Property for Strongly Compact Cardinals

For a cofinal $S \subseteq [\mu]^{<\nu}$, we denote by $[[\, S\,]]^2$ the set of all pairs $(X, Y) \in S \times S$ such that $X \subseteq Y$.

Definition

Let $\lambda \geq \kappa$, the principle $\varphi(\kappa, \lambda)$ establishes that for every $\mu \geq \lambda$ and for every stationary $S \subseteq [\mu]^{<\lambda}$, every $c : [[\, S\,]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H of color $i < \gamma$ which is also stationary, i.e.

$$\text{for every } X, Y \in H \text{ there is } Z \supseteq X, Y \text{ in } H \text{ such that } c(X, Z) = i = c(Y, Z).$$



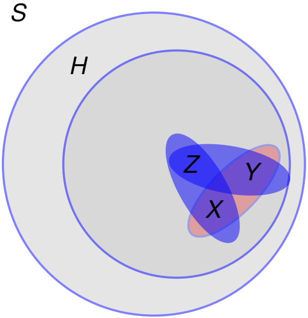
A Partition Property for Strongly Compact Cardinals

For a cofinal $S \subseteq [\mu]^{<\nu}$, we denote by $[[\ S\ ]]^2$ the set of all pairs $(X, Y) \in S \times S$ such that $X \subseteq Y$.

Definition

Let $\lambda \geq \kappa$, the principle $\varphi(\kappa, \lambda)$ establishes that for every $\mu \geq \lambda$ and for every stationary $S \subseteq [\mu]^{<\lambda}$, every $c : [[\ S\ ]]^2 \to \gamma$ with $\gamma < \kappa$ has a quasi homogenous set H of color $i < \gamma$ which is also stationary, i.e.

for every $X, Y \in H$ there is $Z \supseteq X, Y$ in H such that $c(X, Z) = i = c(Y, Z)$.



Theorem

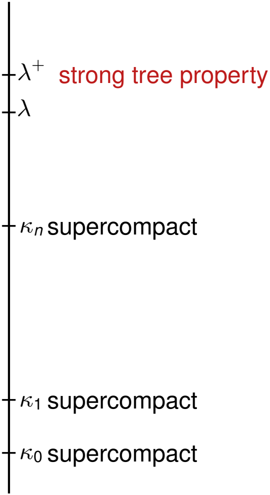
Let κ be a strongly compact cardinal, then $\varphi(\kappa, \lambda)$ holds for every $\lambda \geq \kappa$.

Fontanella - Key Lemma

If $\lambda = lim_{n<\omega}\kappa_n$ where every κ_n satisfies $\varphi(\kappa_n, \lambda^+)$, then then λ^+ satisfies the Strong Tree Property.

Fontanella 2012

If *Cons*(*ZFC* + $\exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then *Cons*(*ZFC* + $\aleph_{\omega+1}$ has the Strong Tree Property).



Fontanella 2012

If $Cons(ZFC + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \aleph_{\omega+1}$ has the Strong Tree Property).

$\lambda^+ = \aleph_{\omega+1}$ strong tree property

$\lambda = \aleph_\omega$

$\kappa_n = \aleph_{n+2}$

$\kappa_1 = \aleph_3$

$\kappa_0 = \aleph_2$

There exists $\nu < \kappa_0$ singular strong limit of cofinality ω, such that

$$Coll(\omega, \nu) \times Coll(\nu^+, < \kappa_0)$$

$$\times \prod_{n<\omega} Coll(\kappa_n, < \kappa_{n+1})$$

forces the strong tree property at $\aleph_{\omega+1}$.

Future work

- Can we find similar characterizations of other large cardinals?
- What cardinals can satisfy those properties?
- How can we use them?

Thank you.