Ergodicity and canonization

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Given two analytic equivalence relations E on X and F on Y, F is E-*generically ergodic* if for every Borel function $f : X \to Y$ which is a homomorphism from E to F, there is a comeager set $C \subseteq X$ such that $f''C$ is contained in one F-class.

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Generic ergodicity is connected with Hjorth's turbulence and is the only known method of showing that an equivalence relation is not classifiable by countable structures.

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Definition

Suppose now that E is an analytic equivalence relation defined on a standard Borel probability space (X, μ). Given an analytic equivalence relation F on Y we say that F is *E-μ-ergodic* if for every Borel function $f : X \to Y$ which is a homomorphism from E to F, there is a measure 1 set $C \subseteq X$ such that $f''C$ is contained in one F-class.

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We will be interested in showing that for certain equivalence relations E, every equivalence relation classifiable by countble structures is E-μ-ergodic.

Definition

Given a Polish space X, a family $\mathbb{P}$ of Borel sets on X and two classes of equivalence relations $\mathbf{E}$ and $\mathbf{F}$ we say that $\mathbf{E}$ *canonizes to* $\mathbf{F}$ w.r.t. $\mathbb{P}$ and write

$$\mathbf{E} \underset{\mathbb{P}}{\to} \mathbf{F}$$

if for every Borel set $B \in \mathbb{P}$ and for every $E \in \mathbf{E}$ there is $C \subseteq B$, $C \in \mathbb{P}$ such that $E \restriction C$ belongs to $\mathbf{F}$.

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Definition

When $\mathbf{F} = \{\mathrm{id}, \mathrm{ev}\}$, then we talk about *total canonization* for $\mathbf{E}$.

Theorem (Silver)

Let $\mathbb{P}$ be the family of perfect subsets X. Then

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Theorem (Connes–Feldman–Weiss)

If X is a Polish space with a Borel probability measure and $\mathbb{P}$ is the family of positive measure sets, then

$$\text{amenable} \underset{\mathbb{P}}{\to} \text{hyperfinite}.$$

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However, as we will see shortly, sometimes ergodicity results for probability measures will use canonization for some other (smaller) I than μ-measure zero sets.

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Theorem 1 (Kanovei–S.–Zapletal)

Suppose I is such that $\mathbb{P}_I$ is proper. If $\mathbb{P}_I$ has total canonization for essentially countable equivalence relations, then $\mathbb{P}_I$ has total canonization for relations classifiable by countable structures.

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Fact

Suppose I is such that $\mathbb{P}_I$ is proper. The following conditions are equivalent:

- $\mathbb{P}_I$ adds a minimal real degree
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The assumption that $\mathbb{P}_I$ adds a minimal real degree means that if $V \subseteq V[y] \subseteq V[G]$ is an intermediate extension given by a real $y \in \mathbb{R}$, then $V[y] = V$ or $V[y] = V[G]$.

Theorem 2 (Kanovei–S.–Zapletal)

Suppose I is such that I is nowhere ccc and $\mathbb{P}_I$ adds a minimal forcing extension. Then $\mathbb{P}_I$ has total canonization for essentially countable equivalence relations.

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The assumption that $\mathbb{P}_I$ adds minimal extension means that there is no model ZFC intermediate between V and $V[G]$ where G is $\mathbb{P}_I$-generic.

Definition

The relations ℓ_p are defined as quotients $\mathbb{R}^\omega/\ell_p$, i.e.

$$x \; \ell_p \; y \quad \text{if} \quad x - y \in \ell_p.$$

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ℓ_1 is the smallest of them and it has a bireducible version on 2^ω, denoted also by E_2 and defined as

$$x \; E_2 \; y \quad \text{if} \quad \sum \{1/(n+1) : x(n) \neq y(n)\} < \infty.$$

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The measure

This way of seeing E_2 displays a measure μ (the Lebesgue measure) on the domain of E_2, which is suitably invariant.

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For every F classifiable by countable structures, F is E_2-μ-ergodic.

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Definition

Given an equivalence relation E and a family $\mathbb{P}$ of Borel sets we say that E is $\mathbb{P}$-*ergodic* if every two sets $B, C \in \mathbb{P}$ there are $x \in B$ and $y \in C$ with $x \; E \; y$.

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There exists a family $\mathbb{P}$ of Borel subsets of 2^ω such that $\mathbb{P} = \mathbb{P}_I$ for some σ-ideal I and

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- E_2 is $\mathbb{P}$-ergodic
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Theorem 3 (ergodicity result) will abstractly follow from the theorem above and canonical Ramsey theory.

Proof of Theorem 3

Suppose $f : 2^\omega \to Y$ is a homomorphism from E_2 to F. Let $E = f^{-1}F$ be the pullback of F to 2^ω.

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By Theorem 2, $\mathbb{P}$ has total canonization for essentially countable equivalence relations and by Theorem 1, this upgrades to classifiable by countable structures.

Thus, E must canonize to a trivial relation on a set in $\mathbb{P}$. We will show that the set has measure 1 and E canonizes to ev.

First observe that E can have at most one class in $\mathbb{P}$. If there were two such E-classes that are in $\mathbb{P}$, then they would contain E_2-related points because E_2 is $\mathbb{P}$-ergodic, and this would contradict $E_2 \subseteq E$.

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Let B be the complement of the single class in $\mathbb{P}$ (if it does not exist, put $B = 2^\omega$). We show that B has measure zero.

Suppose B has positive measure. Wlog assume that B is Borel. Note that $B \in \mathbb{P}_I$ (because I consists of measure zero sets) and on B no E-class belongs to $\mathbb{P}$.

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Now we apply canonical Ramsey theory. Note that on B the E-classes are in I so E cannot canonize to ev on B. Hence by the canonization result, there is $C \subseteq B$ with $C \in \mathbb{P}$ which is E-independent.

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Find two disjoint subsets $C_1, C_2 \subseteq C$, both in $\mathbb{P}$. By E_2-ergodicity, C_1 and C_2 contain E_2-equivalent points, which are thus E-equivalent and contradict canonization. This ends the proof.

Let us comment shortly on Theorem 4. The proof relies heavily on the so-called "concentration of measure" phenomenon (or Gromov's Lévy groups) and a construction similar to that of the Sacks forcing.

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Definition

Given a finitely branching tree T and a family of submeasures ϕ_t for $t \in T$ define the family of *fat trees* as those subtrees $S \subseteq T$ such that

$$\phi_s(\mathrm{succ}_S(s)) \to \infty \qquad \text{over } S.$$

There is a finitely branching tree T and a family of submeasures such that $\mathbb{P}$ equal to the family of fat trees witnesses Theorem 4.

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The fact that E_2 is $\mathbb{P}$-ergodic uses the machinery of concentration of measure.

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One way of constructing the model $V[G]_E$ (if E is countable) is to force with the poset $\mathbb{P}_I^E$ of all I-positive and E-saturated Borel sets.

This intermediate extension gives a decomposition

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Now, if $\mathbb{P}_I$ is nowhere ccc and adds minimal forcing extension, then $\mathbb{Q}$ must be trivial and $\mathbb{P}_I^E$ adds the generic for $\mathbb{P}_I$

The proof is finished by showing that if $\mathbb{P}_I^E$ adds the generic for $\mathbb{P}_I$, then E must canonize to id.