Combinatorial dichotomies and cardinal invariants

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ASL North American annual meeting, University of Waterloo
May 9, 2013

Outline

Calibrating some consequences of PFA

- Cardinal invariants can be used to calibrate certain mathematical statements in the presence of some combinatorial dichotomies.
- These statements are consequences of PFA that contradict CH.

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Prototypical Theorem

Assume $\mathrm{ZFC} + \mathrm{CD}$. *Then the following are equivalent:*

1. $\mathfrak{x} > \omega_1$.
2. ϕ.

Here CD is a combinatorial dichotomy, $\mathfrak{x}$ is some cardinal invariant, and ϕ is some mathematical statement.

Two recent examples

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Theorem (Todorcevic and Torres-Perez)

Assume ZFC + *Rado's Conjecture* (RC)*. Then the following are equivalent:*

1. $\mathfrak{c} > \omega_1$
2. *There are no special* ω_2*-Aronszajn trees.*

Two recent examples

Theorem (Brech and Todorcevic)

Assume ZFC + *P-Ideal Dichotomy* (PID)*. Then the following are equivalent:*

1. $\mathfrak{b} > \omega_1$.
2. *Every non-separable Asplund space has an uncountable bi-orthogonal system.*

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2. *Every non-separable Asplund space has an uncountable bi-orthogonal system.*

- In these examples PID and RC are the combinatorial dichotomies and $\mathfrak{c}$ and $\mathfrak{b}$ are the cardinal invariants.
- This talk will mostly focus on PID.

P ideals

Definition

Let X be an uncountable set. An ideal $\mathcal{I} \subset [X]^{\leq\omega}$ is called a P-ideal if for every countable collection $\{a_n : n \in \omega\} \subset \mathcal{I}$, there is $a \in \mathcal{I}$ such that $\forall n \in \omega\, [a_n \subset^* a]$.
Here $a \subset^* b$ means $a \setminus b$ is finite.
All ideals are assumed to be non-principal, meaning that $[X]^{<\omega} \subset \mathcal{I}$.

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Example

Let X be an uncountable set and let Y be an uncountable subset of X.
$\mathcal{I} = [X]^{<\omega} \cup [Y]^{\leq\omega}$.

P-Ideal Dichotomy

Example

Let X be an uncountable set. Let $\{X_n : n \in \omega\}$ be a collection of uncountable subset of X. Put $\mathcal{I} = \left\{a \in [X]^{\leq\omega} : \forall n \in \omega \left[|a \cap X_n| < \omega\right]\right\}$

P-Ideal Dichotomy

Definition

The P-ideal dichotomy (PID) *is the following statement: For any P-ideal $\mathcal{I}$ on an uncountable set X either*

(1) *There is an uncountable set $Y \subset X$ such that $[Y]^{\leq\omega} \subset \mathcal{I}$ (so $\mathcal{I}$ contains a copy of the first example)*

or

(2) *There exist $\{X_n : n \in \omega\}$ such that the X_n are pairwise disjoint, $X = \bigcup_{n\in\omega} X_n$, and $\forall n \in \omega \, [[X_n]^{\omega} \cap \mathcal{I} = 0]$ (so $\mathcal{I}$ is contained in a copy of the second example).*

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- Suslin's Hypothesis (Todorcevic).
- Singular Cardinals Hypothesis (Viale).
- $\square_{\kappa,\omega}$ fails for all uncountable cardinals κ (R.).

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How does PID influence statements that contradict CH?

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How does PID influence statements that contradict CH?

Answer: It tends to push many such statements down to combinatorial questions about sets of reals.

Cardinal Invariants

- For functions $f, g \in \omega^\omega$, $f <^* g$ means $\forall^\infty n \in \omega\, [f(n) < g(n)]$.
- A set $F \subset \omega^\omega$ is said to be *unbounded* if there is no $g \in \omega^\omega$ such that $\forall f \in F\, [f <^* g]$.

Definition

$\mathfrak{b} = \min \{|F| : F \subset \omega^\omega \wedge F \textit{ is unbounded}\}$

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- A family $F \subset [\omega]^\omega$ is said to have the *finite intersection property (FIP)* if for any $a_0, \ldots, a_k \in F$, $a_0 \cap \cdots \cap a_k$ is infinite.

Definition

$\mathfrak{p} = \min \left\{|F| : F \subset [\omega]^\omega \wedge F \textit{ has the FIP } \wedge \neg \exists b \in [\omega]^\omega \, \forall a \in F \, [b \subset^* a]\right\}$

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- It is easy to show $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{c}$.
- Also $\mathfrak{p} \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{c}$.
- $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{M})$ are independent.

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- Also $\mathfrak{p} \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{c}$.
- $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{M})$ are independent.
- PID + $\mathfrak{p} > \omega_1$ implies many of the consequences of PFA that contradict CH.

An example

Definition

A regular hereditarily separable space that is not Lindelöf is called an S-Space.

Theorem (Todorcevic)

Assume $\mathrm{PID} + \mathfrak{p} > \omega_1$*. Then there are no S Spaces. Moreover, if* $\mathfrak{b} = \omega_1$*, then there is a (first countable) S Space.*

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- There is a “gap” between $\mathfrak{b}$ and $\mathfrak{p}$.
- There are other examples like this. We will see 2 in a bit.

The project

General Problem 1

Given a statement ϕ *which is a consequence of* $\mathrm{PID} + \mathrm{MA}_{\aleph_1}$*, find a cardinal invariant* $\mathfrak{x}$ *such that* ϕ *is equivalent to* $\mathfrak{x} > \omega_1$*.*

A slightly less ambitious project is

General Problem 2

Given a statement ϕ *which is a consequence of* $\mathrm{PID} + \mathfrak{p} > \omega_1$*, investigate whether* ϕ *is equivalent to* $\mathfrak{p} > \omega_1$

Why do this?

- Allows one to calibrate the relative strength of various consequences of PFA over $\mathrm{ZFC} + \mathrm{PID}$.

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- Allows one to calibrate the relative strength of various consequences of PFA over $\mathrm{ZFC} + \mathrm{PID}$.
- One often needs to find sharper proofs.
- Decomposes the influence of PFA on ϕ into two parts: A part that is consistent with CH + the essential combinatorial bit contradicting CH captured by the cardinal invariant.

A model for General Problem 2

Definition

Let $\mathbb{S}$ be a coherent Suslin tree. $\mathrm{PFA}(\mathbb{S})$ *is the following statement. If $\mathbb{P}$ is a poset which is proper and preserves $\mathbb{S}$ and $\{D_\alpha : \alpha < \omega_1\}$ is a collection of dense subsets of $\mathbb{P}$, then there is a filter G on $\mathbb{P}$ such that $\forall \alpha < \omega_1 \, [G \cap D_\alpha \neq 0]$.*

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- $\mathrm{PFA}(\mathbb{S})$ says that the maximal amount of PFA compatible with the existence of $\mathbb{S}$ holds.
- After forcing with $\mathbb{S}$ over a model of $\mathrm{PFA}(\mathbb{S})$ many consequences of PFA hold.

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- $\mathrm{PFA}(\mathbb{S})$ says that the maximal amount of PFA compatible with the existence of $\mathbb{S}$ holds.
- After forcing with $\mathbb{S}$ over a model of $\mathrm{PFA}(\mathbb{S})$ many consequences of PFA hold.
- In particular, after forcing with $\mathbb{S}$, PID holds.
- Also after forcing with $\mathbb{S}$, *most* cardinal invariants are greater than ω_1. For example, $\mathfrak{b} > \omega_1$ and $\mathrm{cov}(\mathcal{M}) > \omega_1$.

A model for General Problem 2

- However $\mathfrak{p} = \omega_1$.
- So after forcing with $\mathbb{S}$ over a model of $\mathrm{PFA}(\mathbb{S})$, one gets almost all of the consequences of PFA that are consistent with $\mathrm{PID} + \mathfrak{p} = \omega_1$.
- So if ϕ some consequence of PFA which you suspect *is not equivalent* to $\mathfrak{p} > \omega_1$ over $\mathrm{ZFC} + \mathrm{PID}$, then this model is a good place to look.

5 cofinal types

Definition

Given two (upward) directed posets $\mathbb{P}$ and $\mathbb{Q}$, we say that a map $f : \mathbb{P} \to \mathbb{Q}$ is a Tukey map if it maps (upward) unbounded sets in $\mathbb{P}$ to unbounded sets in $\mathbb{Q}$. We say that a map $g : \mathbb{Q} \to \mathbb{P}$ is a convergent map if the image of every (upward) cofinal subset of $\mathbb{Q}$ is cofinal in $\mathbb{P}$.

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- Fact: There is a Tukey map $f : \mathbb{P} \to \mathbb{Q}$ iff there is a convergent $g : \mathbb{Q} \to \mathbb{P}$

Definition

We say $\mathbb{P}$ is Tukey reducible to $\mathbb{Q}$ and we write $\mathbb{P} \leq_T \mathbb{Q}$ if there is a a Tukey map $f : \mathbb{P} \to \mathbb{Q}$.

5 cofinal types

- We have a natural equivalence $\mathbb{P} \equiv_T \mathbb{Q}$ iff $\mathbb{P} \leq_T \mathbb{Q}$ and $\mathbb{Q} \leq_T \mathbb{P}$. Then we say $\mathbb{P}$ and $\mathbb{Q}$ are *Tukey equivalent* or have the same *Tukey type*.

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- Fact: $\mathbb{P} \equiv_T \mathbb{Q}$ iff both $\mathbb{P}$ and $\mathbb{Q}$ embed a cofinal subsets of another directed set $\mathbb{R}$.

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- Fact: $\mathbb{P} \equiv_T \mathbb{Q}$ iff both $\mathbb{P}$ and $\mathbb{Q}$ embed a cofinal subsets of another directed set $\mathbb{R}$.

Theorem (Todorcevic[2])

Under PFA *there are only 5 Tukey types of size at most* $\aleph_1$*:* 1, ω, ω_1, $\omega \times \omega_1$, $[\omega_1]^{<\omega}$. *In fact,* $\mathrm{PID} + \mathfrak{p} > \omega_1$ *implies this.*

5 cofinal types

Definition

cof($\mathcal{F}_\sigma$) *is the least* κ *such that there exists a tall* F_σ *ideal* $\mathcal{I}$ *on* ω *and a directed cofinal* $X \subset \mathcal{I}$ *(i.e.* X *is cofinal in* $\langle \mathcal{I}, \subset \rangle$*) such that* $|X| = \kappa$*.*

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- cof($\mathcal{F}_\sigma$) has been investigated by Hrušák and Zapletal [1].
- It is not hard to see that cov($\mathcal{M}$) $\leq$ cof($\mathcal{F}_\sigma$).

5 cofinal types

Theorem (R. and Todorcevic)

Assume PID*. The following are equivalent.*

1. $\min\{\mathfrak{b}, \mathrm{cof}(\mathcal{F}_\sigma)\} > \omega_1$.
2. 1, ω, ω_1, $\omega \times \omega_1$, *and* $[\omega_1]^{<\omega}$ *are the only cofinal types of directed sets of size at most* $\aleph_1$.

Remark

$\mathfrak{b}$ *and* $\mathrm{cof}(\mathcal{F}_\sigma)$ *are independent even over* ZFC + PID*. So one cannot really simplify (1) above.*

A strong version of Dushnik-Miller Theorem

Definition

For an ordinal α, $\omega_1 \to (\omega_1, \alpha)^2$ *means that for any* $c : [\omega_1]^2 \to 2$ *either there exists* $X \in [\omega_1]^{\omega_1}$ *such that* $c''[X]^2 = \{0\}$, *or there exists* $X \subset \omega_1$ *with* $\mathrm{otp}(X) = \alpha$ *and* $c''[X]^2 = \{1\}$.

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- $\omega_1 \to (\omega_1, \omega + 1)^2$ is a theorem of ZFC (Dushnik-Miller)
- $\omega_1 \to (\omega_1, \omega_1)^2$ is false (Sierpinski).

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Theorem (Todorcevic)

PFA *implies that $\omega_1 \to (\omega_1, \alpha)^2$, for every $\alpha < \omega_1$. In fact, this follows from* $\mathrm{PID} + \mathfrak{p} > \omega_1$*. Moreover, if $\mathfrak{b} = \omega_1$, then $\omega_1 \not\to (\omega_1, \omega + 2)^2$.*

A strong version of Dushnik-Miller Theorem

Theorem (R. and Todorcevic)

$\mathrm{PFA}(\mathbb{S})$ *implies that the coherent Suslin tree* $\mathbb{S}$ *forces* $\omega_1 \to (\omega_1, \alpha)^2$ *to hold for all* $\alpha < \omega_1$.

A strong version of Dushnik-Miller Theorem

Definition

For any $A \subset \mathbb{S}$, $A^{[2]} = \{\{a, b\} : a, b \in A$ *and* $a < b\}$. *For* $t \in \mathbb{S}$, $\mathrm{pred}(t)$ *denotes the set of predecessors of* t*, that is* $\{s \in \mathbb{S} : s \leq t\}$. *For a set* X *and* $t \in \mathbb{S}$, $\mathrm{pred}_X(t) = \mathrm{pred}(t) \cap X$.

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We also consider the following variation of $A^{[2]}$:

Definition

Let $Y \subset \mathbb{S}$ *and* $g : Y \to \mathbb{S}$. *Then* $Y_g^{[2]}$ *denotes* $\{\{a, b\} : a, b \in Y \text{ and } a < b \text{ and } g(a) \leq b\}$.

A strong version of Dushnik-Miller Theorem

Definition

If $S \subset \mathbb{S}$ and $c : S^{[2]} \to 2$ is a coloring, then
$K_i = \left\{\{s, t\} \in S^{[2]} : c\left(\{s, t\}\right) = i\right\}$, *for any* $i \in \{0, 1\}$

A strong version of Dushnik-Miller Theorem

Theorem

Assume $\mathrm{PFA}(\mathbb{S})$. *Let* $S \in [\mathbb{S}]^{\omega_1}$ *and* $c : S^{[2]} \to 2$. *Then either there exist* $Y \in [S]^{\omega_1}$ *and* $g : Y \to \mathbb{S}$ *such that* $\forall y \in Y \, [g(y) \geq y]$ *and* $Y_g^{[2]} \subset K_0$ *or for each* $\alpha < \omega_1$, *there exists* $s \in S$ *and* $B \subset \mathrm{pred}_S(s)$ *such that* $\mathrm{otp}(B) = \alpha$ *and* $B^{[2]} \subset K_1$.

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Theorem

Let $\mathbb{S}$ *be a (coherent) Suslin tree. There is* $c : \mathbb{S}^{[2]} \to 2$ *such that*

1. *There is no* $X \in [\mathbb{S}]^{\omega_1}$ *such that* $X^{[2]} \subset K_0$.
2. *There is no* $s \in \mathbb{S}$ *and* $B \subset \mathrm{pred}_{\mathbb{S}}(s)$ *such that* $\mathrm{otp}(B) = \omega^2$ *and* $B^{[2]} \subset K_1$.

Questions

Question

Can we find a cardinal invariant $\mathfrak{x}$ *such that* $\mathfrak{x} > \omega_1$ *is equivalent over* $\mathrm{ZFC} + \mathrm{PID}$ *to* $\omega_1 \rightarrow (\omega_1, \alpha)^2$ *for all* $\alpha < \omega_1$*?*

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Question

Can we find a cardinal invariant $\mathfrak{x}$ *such that* $\mathfrak{x} > \omega_1$ *is equivalent over* $\mathrm{ZFC} + \mathrm{PID}$ *to the statement that there are no S spaces?*

Question

Is there an S space after forcing with the coherent Suslin tree $\mathbb{S}$ *over a model of* $\mathrm{PFA}(\mathbb{S})$*?*

Bibliography

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