**Definition**

A **Hilbert space** is a vector space $H$ together with a complete inner product $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{C}$.

- $H = \ell^2 = \{(x_n) \subseteq \mathbb{C} : \sum |x_n|^2 < \infty\}; \langle (x_n), (y_n) \rangle = \sum x_n \overline{y_n}$.
- $\ell^2 \approx$ quantum analog of $\omega$.
- The set of bounded linear operators $\mathcal{B}(H)$ on $H$ is a $C^*$-algebra, i.e. a Banach algebra with involution $^*$ satisfying $||xx^*|| = ||x||^2$, ($^*$ is the adjoint, i.e. $\langle Tv, w \rangle = \langle v, T^*w \rangle$).

**Definition**

$P \in \mathcal{B}(H)$ is a **projection** if $P = P^* = P^2$. Denote by $\mathcal{P}(\mathcal{B}(H))$.

- Projections correspond to closed subspaces via $P \mapsto \mathcal{R}(P)$ and $PQ = P = QP \iff \mathcal{R}(P) \subseteq \mathcal{R}(Q)$.
- $\mathcal{P}(\mathcal{B}(H)) \approx$ quantum/non-commutative analog of $\mathcal{P}(\omega)$.
Compact Operators

**Definition**

$T \in \mathcal{B}(H)$ is *compact* if $\overline{T[B_1(H)]}$ is compact. Denote by $\mathcal{K}(H)$.

- $\mathcal{K}(H) = \{ T \in \mathcal{B}(H) : \dim(\mathcal{R}(T)) < \infty \}$.
- $\mathcal{P}(\mathcal{K}(H)) = \{ P \in \mathcal{P}(\mathcal{B})(H) : \dim(\mathcal{R}(T)) < \infty \}$.
- $\mathcal{P}(\mathcal{K}(H)) \approx$ non-commutative analog of $\text{Fin} = [\omega]^{<\omega}$.
- $\mathcal{K}(H)$ is an ideal in $\mathcal{B}(H)$.
- $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is a C*-algebra, the *Calkin Algebra*.
- $\mathcal{P}(\mathcal{C}(H)) \approx$ non-commutative analog of $\mathcal{P}(\omega)/\text{Fin}$. 
• \( \mathcal{P}(\mathcal{C}(H)) \) is separative (\( \mathcal{C}(H) \) has real rank zero).

• Unlike \( \mathcal{P}(\omega)/\text{Fin} \), \( \mathcal{P}(\mathcal{C}(H)) \) is \textit{not} a lattice.

\[
\exists p \land q \iff \sup(\sigma(pq) \setminus \{1\}) < 1 \iff \exists p \lor q.
\]

• So maximal centred \( \nleftrightarrow \) maximal (downwards) directed.

\textbf{Theorem (B. 2010)}

Any \( (p_n) \subseteq \mathcal{P}(\mathcal{C}(H)) \) has \( \leq \)-equivalent decreasing \( (q_n) \subseteq \mathcal{P}(\mathcal{C}(H)) \).

• \((\omega,\omega)\)-pregaps in \( \mathcal{P}(\mathcal{C}(H)) \) lift to \( \mathcal{P}(\mathcal{B}(H)) \) (real rank zero).

• \( \mathcal{P}(\mathcal{B}(H)) \) is a complete lattice so no \((\omega,\omega)\)-gaps.

\textbf{Corollary}

No non-trivial finite or countable gaps in \( \mathcal{P}(\mathcal{C}(H)) \).
Take a basis \((e_n)_{n \in \omega} \subseteq H\). For \(A \subseteq \omega\), define \(P_A \in \mathcal{P}(\mathcal{B}(H))\) by \(\mathcal{R}(P_A) = \text{span}(e_n)_{n \in A}\). Then

\[ A \subseteq B \iff P_A \leq P_B \quad \text{and} \quad A \subseteq^* B \iff \pi(P_A) \leq \pi(P_B), \]

so \(\mathcal{P}(\omega)\) and \(\mathcal{P}(\omega)/\text{Fin}\) embed in \(\mathcal{P}(\mathcal{B}(H))\) and \(\mathcal{P}(\mathcal{C}(H))\) resp.

Also, \(A \mapsto P_A\) is continuous w.r.t. weak operator topology.

**Question**

For what other ideals \(I\) does \(\mathcal{P}(\omega)/I\) embed in \(\mathcal{P}(\mathcal{C}(H))\)? What if we require the embedding to have a continuous lifting?
### Gap Spectrum

#### Theorem (Steprans)
For any $p \in \mathcal{P}(C(H))$, $\{A \subseteq \omega : \pi(P_A) \leq p\}$ is an analytic $p$-ideal.

#### Theorem (Todorcevic/Solecki)
The orthogonal of an analytic $p$-ideal is countably generated.

#### Corollary
The map $A \mapsto \pi(P_A)$ is gap preserving.

#### Theorem (Zamora-Aviles 2009)
$\mathcal{P}(C(H))$ contains an analytic Hausdorff gap.

- Under (MA) this is a $(c, c)$ gap.
- Under (TA) $\mathcal{P}(\omega)/\text{Fin}$ has only $(\omega_1, \omega_1)$ and $(\omega, b)$ gaps.
- $(\text{MA}) + (\text{TA}) + (c = \omega_2)$, $\text{spec}(\mathcal{P}(\omega)/\text{Fin}) \subset \not\subseteq \text{spec}(\mathcal{P}(C(H)))$. 

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Automorphisms

- Any (almost) 1-1 onto $f : \omega \to \omega$ yields a trivial automorphism on $\mathcal{P}(\omega)/\text{Fin}$ by $A \mapsto f(A)$.
- Any unitary $U \in C(H)$ ($U^*U = UU^* = 1$) yields an inner automorphism on $C(H)$ by $T \mapsto UTU^*$ ($R(P) \mapsto U[R(P)]$).
- $\mathcal{P}(\omega)/\text{Fin}$ has non-trivial automorphisms under CH (Rudin) and none under PFA (Shelah-Steprans).

**Theorem (Phillips-Weaver 2007)**

Under CH, $C(H)$ has outer automorphisms.

- Unknown for larger $H$ but same for some higher dimension Calkin-like algebras (Farah-McKenney-Schimmerling).

**Theorem (Farah 2010)**

Under TA, all automorphisms of $C(H)$ are inner.

- Same for larger $H$ under PFA (Farah).
• $\mathcal{P}(\omega)/\text{Fin}$ cardinal invariants have ($\geq$)2 analogs ∴:

\[
p \leq q \quad \Rightarrow \quad p \land q^\perp = 0
\]
\[
\iff \quad \iff
\]
\[
pq^\perp = 0 \quad \|pq^\perp\| < 1
\]

• $p$ strongly splits $q$ \iff $p \land q \neq 0 \neq p^\perp \land q$.

• $p$ weakly splits $q$ \iff $p \land q \neq 0$ and $q \not\leq p$.

$s^\perp = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(C(H)) \text{ is a strongly splitting family}\}$.

$s^* = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(C(H)) \text{ is a weakly splitting family}\}$. 

Tristan Bice

Set Theory and $C^*$-algebras
V is a block subspace if \( \exists \) IP \( (I_n) \) and \( \exists \) \( (v_n) \) \( \subseteq H \) s.t. 
\[ V = \text{span}(v_n) \text{ and } \forall n (v_n \in \text{span}\{e_k : k \in I_n\}). \]

Block subspaces are \( \leq^* \)-dense.

Given \( \text{inf dim } V \subseteq H \) recursively pick unit vectors \( (v_n) \subseteq V \)

\[ v_0 = (0, \frac{1}{5}, \frac{3}{4}, \frac{1}{2}, \frac{1}{10}, \ldots) \text{ (arbitrary)} \]

\[ v_1 = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{16}}, \ldots) \in V \cap \ell^2_{k_0}, \ k_0 \gg 0 \]

\[ v_2 = (0, 0, 0, 0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots) \in V \cap \ell^2_{k_1}, \ k_1 \gg k_0 \]

\[ \vdots \]

\[ V \supseteq \text{span}(v_n) =^* \text{ block subspace. } \square \]
Interval Paritions

- So card invs on $\mathcal{P}(\mathcal{C}(H))$ often related to IP card invs.
- Eg. $A \subseteq \omega$ splits IP $(I_n) \iff \exists \infty n I_n \subseteq A \text{ and } \exists \infty n I_n \subseteq \omega \setminus A$.

$\text{s}_{\text{IP}} = \min\{|A| : A \subseteq \mathcal{P}(\omega) \text{ is an IP splitting family}\}$.

$A \subseteq \mathcal{P}(\omega)$ IP splitting $\Rightarrow (\pi(P_A))_{A \in \mathcal{A}}$ strongly splitting.

$\Rightarrow s^\perp \leq s_{\text{IP}}$.

$\text{s}_{\text{IP}} = \max(s, b)$ (A. Kamburelis, B. Weglorz (1995)).

$t \leq s^* \leq s^\perp \leq \max(s, b)$
• $t = b \Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ generating a non-meagre p-filter.
  \[\Rightarrow (\pi(P_{A_\xi}))\text{ is a tower in } \mathcal{P}(C(H))\]
• Given IP $(I_n)$ take $v_n = \sum_{k \in I_n} e_k$ and $\mathcal{R}(P) = \overline{\text{span}}(v_n)$.
• If $A \subseteq \omega$ s.t. $|A \cap I_n|/|I_n| \to 1$ then $\pi(P) \leq \pi(P_A)$.
• $\sup_n(m(\sigma\text{-}n\text{-}linked))) = \text{non}(\mathcal{M}) \Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ s.t.
  \[\forall \xi |A_\xi \cap I_n|/|I_n| \to 1 \Rightarrow (\pi(P_{A_\xi}))\text{ is not a tower in } \mathcal{P}(C(H))\].

**Theorem (Brendle)**
Consistently no towers in $\mathcal{I}^*$ for any analytic p-ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$.

**Corollary**
Consistently all towers in $[\omega]^\omega$ remain towers in $\mathcal{P}(C(H))$.

• $t^* = t^\perp \geq t$ (B., Wofsey, Bell, Malliaris-Shelah).
Other Cardinal Invariants

- $b^* = b^\perp = b$ and $a^* = a^\perp = a$ (Zamora-Aviles).
- $b \leq a^\perp$ (Brendle). Consistently $a^\perp = a = \aleph_1 < c$ (finite conditions) and $a^\perp = a = c > \aleph_1$ (MA) (Wofsey).
- In the Sacks model $a^* = \aleph_1 < c$ (B.). Is $a^* = \aleph_1$?
States to Subsets

Let $\phi$ be a state on a C*-algebra $A$. Then

- $\{ T \in A : \phi(T^* T) = 0 \}$ is a closed left ideal.
- $\{ T \in A_+ : \phi(T) = 0 \}$ is a closed hereditary cone.
- $\{ T \in A_+^1 : \phi(T) = 1 \}$ is a norm filter.

**Definition [B. (2011)]**

$F \subseteq A_+^1$ is a norm filter if, whenever $T \in A_+^1$ and

$$\inf\{ \|(1 - T)S_1 \ldots S_n\| : n \in \omega \text{ and } S_1, \ldots, S_n \in F \} = 0,$$

we necessarily have $T \in F$. 
Let $\phi$ be a pure state on a C*-algebra $A$. Then

- $\{ T \in A : \phi(T^* T) = 0 \}$ is a maximal left ideal.
- $\{ T \in A_+ : \phi(T) = 0 \}$ is a maximal hereditary cone.
- $\{ T \in A_+^1 : \phi(T) = 1 \}$ is maximal norm centred.

**Definition [Farah and Weaver (~2009)]**

$F \subseteq A_+^1$ is a *norm centred* if $\|S_1 \ldots S_n\| = 1$, for $S_1, \ldots, S_n \in F$.

- $F$ is a (proper) norm filter $\nRightarrow F$ is norm centred.
- $F$ is a maximal norm filter $\Leftrightarrow F$ is maximal norm centred.
- The above yields a one-to-one correspondence.
The Real Rank Zero Case

- If $A$ has RR0 then $\phi \mapsto \{ P \in \mathcal{P}(A) : \phi(P) = 1 \}$ takes pure states to maximal norm centred subsets and vice versa.
- If $\mathcal{P}(A) \setminus \{0\}$ is also $\sigma$-closed, i.e. decreasing sequences are bounded, (e.g. if $A = C(H)$ = the Calkin algebra) then
  \[
  F \text{ is norm centred} \iff F \text{ is centred},
  \]
i.e. finite subsets of $F$ have lower bounds in $\mathcal{P}(A) \setminus \{0\}$.

Conjecture [Kadison-Singer (1959)]

Pure states on atomic MASAs of $\mathcal{B}(H)$ extend uniquely.

- KS $\iff$ whenever $\mathcal{U}$ is an ultrafilter on $\omega$, $\{ \pi(P_U) : U \in \mathcal{U} \}$ has a unique maximal centred extension in $\mathcal{P}(C(H))$. 

The Kadison-Singer Conjecture

- True for Q-points [Reid (1970)].
- True for rapid P-points [B. (2011)].

Definition (special ultrafilters)

An ultrafilter $\mathcal{U}$ on $\omega$ is

- a **Q-point** if, whenever $(I_n)$ is a partition of $\omega$ into finite intervals, we have $U \in \mathcal{U}$ with $|U \cap I_n| \leq 1$, for all $n$.
- **rapid** if, whenever $f \in \omega^\omega$, we have $U \in \mathcal{U}$ with $|U \cap [1, f(n)]| \leq n$, for all $n$.
- a **P-point** if $\mathcal{U}$ is $\sigma$-closed w.r.t. $\subseteq^*$. ($X \subseteq^* Y \iff X \setminus Y$ is finite)

Q-points are rapid but not vice versa.

Under MA(countable)$\iff$CH, there are rapid P-points that are not Q-points (Flaskova ($\sim$2010)).
Paving Conjectures

\[ \iff \text{KS [Anderson (1980)]} \]

Given \( \epsilon > 0 \) and \( P \in \mathcal{P}(\mathcal{B}(H)) \) we have a partition \( X_1, \ldots, X_n \) of \( \omega \) s.t. \( \|PP_{X_m}\|^2 + \|P^\perp P_{X_m}\|^2 \leq 1 + \epsilon \), for all \( m < n \).

\[ \iff \text{KS [Akemann and Anderson (1991), Weaver (2004), B. (2011)]} \]

There exists \( \delta > 0 \) such that for \( P \in \mathcal{P}(\mathcal{B}(H)) \) satisfying \( \langle Pe_k, e_k \rangle < \delta \), for all \( k \), we have a partition \( X_1, \ldots, X_n \) of \( \omega \) s.t. for all \( m < n \), either \( \|PP_{X_m}\| < 1 \) or \( \|P^\perp P_{X_m}\| < 1 \).

\[ \text{Theorem [B. (2011)]} \]

Assume \( A \) is a RR0 C*-algebra, \( \mathcal{F} \subseteq \mathcal{P}(A) \) is norm centred and \( \mathcal{M} \subseteq \mathcal{P}(A) \) is a maximal extension of \( \mathcal{F} \). Then either \( \mathcal{M} \) is unique or we have \( P \in \mathcal{M} \) such that \( \mathcal{F} \cup \{P^\perp\} \) is norm centred.
Theorem [B. (2011)]
If \( \mathcal{U} \) is a P-point and \( \text{KS}(\mathcal{U}) \) holds then the upwards closure \( \mathcal{P} \supseteq \pi[\mathcal{P}_\mathcal{U}] \) is maximal centred.

- Hence \( \mathcal{P} \) is directed, i.e. given \( p, q \in \mathcal{C}(H) \) with \( \phi_\mathcal{U}(p) = \phi_\mathcal{U}(q) = 1 \), there exists \( r \leq p, q \) with \( \phi_\mathcal{U}(r) = 1 \).

Theorem [B. (2011)]
If \( \mathcal{U} \) is a non-P-point then \( \exists p, q \in \mathcal{P}(\mathcal{C}(H)) \) with \( \phi(p) = \phi(q) = 1 \) and \( \phi(r) = 0 \) whenever \( r \leq p, q \) and \( \phi \) is a state: \( \phi[\pi[\mathcal{P}_\mathcal{U}]] = \{1\} \).

Theorem [B. (2011)]
Consistently (with ZFC), \( \forall \) pure state \( \phi \) on \( \mathcal{C}(H) \), \( \exists p, q \in \mathcal{P}(\mathcal{C}(H)) \) with \( \phi(p) = \phi(q) = 1 \) and \( \sup\{\phi(r) : r \leq p, q\} < 1 \).
Ultrafilters of Projections

Question

For ultrafilter $\mathcal{U}$ on $\omega$, does $\pi[P_\mathcal{U}]$ have a unique ultrafilter extension?

- Yes for Q-points. In fact, $\pi[P_\mathcal{U}]$ is an ultrafilter base.
- Yes for rapid P-points. Again, $\pi[P_\mathcal{U}]$ is an ultrafilter base.
- Not always an ultrafilter base: Take interval partition $(I_n)$ of $\omega$ with $|I_n| \to \infty$ and ultrafilter $\mathcal{U}$ with $\limsup |U \cap I_n|/|I_n| > 0$, for all $U \in \mathcal{U}$. Take $P$ with $\mathcal{R}(P) = \overline{\text{span}}\{\sum_{m \in I_n} e_m : n \in \omega\}$. Then $\pi(P_U) \not\subseteq \pi(P)$ but $\dim(\mathcal{R}(P_U) \cap \mathcal{R}(P)) = \infty$, for $U \in \mathcal{U}$. So $\{\pi(Q) : Q \in \mathcal{P}(\mathcal{B}(H)) \land \exists U \in \mathcal{U}(\mathcal{R}(P_U) \cap \mathcal{R}(P) \subseteq \mathcal{R}(Q))\}$ is a filter properly containing the upwards closure of $\pi[P_\mathcal{U}]$. □
Forcing: $\mathcal{P}(\omega)/\text{Fin}$ vs $\mathcal{P}(\mathcal{C}(H))$

- $\mathcal{P}(\omega)/\text{Fin}$ is $\sigma$-closed so forcing with it adds no reals.
- But it does add a canonical subset of reals, namely a selective ultrafilter $\Leftrightarrow$ simultaneously a $P$-point and a $Q$-point.
- Likewise $\mathcal{P}(\mathcal{C}(H))$ is $\sigma$-closed and adds a ‘selective’ maximal centred filter $\mathcal{P}$, i.e. $\mathcal{P}$ is $\sigma$-closed and, for any basis $(e_n)$ of $H$ and finite interval partition $(I_n)$ of $\omega$, there exists a coarser partition $(J_n)$ and $(v_n) \subseteq H$ with $v_n \in \text{span}_{k \in J_n}(e_k)$ such that $\pi(P) \in \mathcal{P}$, where $P$ is the projection onto $\text{span}(v_n)$.

Questions

What other similarities do these forcing extensions have? Could they even be the same? Do (e.g. Ramsey type) theorems about selective ultrafilters in $\mathcal{P}(\omega)/\text{Fin}$ have natural analogs for these ‘selective’ maximal centred filters in $\mathcal{P}(\mathcal{C}(H))$? What about similar analogous forcing notions (e.g. Mathias forcing)?