Induced Ramsey theorems for clopen graphs

Stefan Geschke (joint work with Stefanie Frick)

October 26, 2012

Ramsey theory in the finite, countable, and on the reals

Plain Ramsey theorems

Theorem

1) (Ramsey) For all $m \in \omega$ and $d \in \omega \setminus \{0\}$ there is $n \in \omega$ such that

$$n \to (m)^d_2.$$

2) (Ramsey) For all $d \in \omega \setminus \{0\}$,

$$\omega \to (\omega)^d_2.$$

3) (Galvin; Blass) For all $d \in \omega \setminus \{0\}$ there is a finite partition

$$[2^\omega]^d = C_1 \dot\cup \dots \dot\cup C_n$$

such that for all Baire-measurable colorings $c : [2^\omega]^d \to 2$ there is a non-empty perfect set $P \subseteq 2^\omega$ such that c is constant on $[P]^d \cap C_i$ for all $i \in \{1, \dots, n\}$.

Plain Ramsey theorems

Theorem

1) (Ramsey) For all $m \in \omega$ *and* $d \in \omega \setminus \{0\}$ *there is* $n \in \omega$ *such that*

$$n \to (m)^d_2.$$

2) (Ramsey) For all $d \in \omega \setminus \{0\}$,

$$\omega \to (\omega)^d_2.$$

3) (Galvin; Blass) For all $d \in \omega \setminus \{0\}$ *there is a finite partition*

$$[2^\omega]^d = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for all Baire-measurable colorings $c : [2^\omega]^d \to 2$ *there is a non-empty perfect set* $P \subseteq 2^\omega$ *such that* c *is constant on* $[P]^d \cap C_i$ *for all* $i \in \{1, \ldots, n\}$.

Plain Ramsey theorems

Theorem

1) (Ramsey) For all $m \in \omega$ and $d \in \omega \setminus \{0\}$ there is $n \in \omega$ such that

$$n \to (m)^d_2.$$

2) (Ramsey) For all $d \in \omega \setminus \{0\}$,

$$\omega \to (\omega)^d_2.$$

3) (Galvin; Blass) For all $d \in \omega \setminus \{0\}$ there is a finite partition

$$[2^\omega]^d = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for all Baire-measurable colorings $c : [2^\omega]^d \to 2$ there is a non-empty perfect set $P \subseteq 2^\omega$ such that c is constant on $[P]^d \cap C_i$ for all $i \in \{1, \ldots, n\}$.

We now consider results with infinite d.

Theorem

1) (Galvin-Prikry; Mathias; Silver) For every partition $[\omega]^\omega = K_0 \dot\cup K_1$ with K_0 analytic there is $H \in [\omega]^\omega$ such that for some $i \in 2$,

$$[H]^\omega \subseteq K_i.$$

2) (Louveau-Shelah-Veličković) For every partition $K_0 \dot\cup K_1$ of the set of **strongly increasing sequences** *in 2^ω with K_0 analytic there is a perfect set P such that all strongly increasing sequences of P are in the same part K_i of the partition.*

We now consider results with infinite d.

Theorem

1) (Galvin-Prikry; Mathias; Silver) For every partition $[\omega]^\omega = K_0 \dot\cup K_1$ with K_0 analytic there is $H \in [\omega]^\omega$ such that for some $i \in 2$,

$$[H]^\omega \subseteq K_i.$$

2) (Louveau-Shelah-Veličković) For every partition $K_0 \dot\cup K_1$ of the set of **strongly increasing sequences** *in 2^ω with K_0 analytic there is a perfect set P such that all strongly increasing sequences of P are in the same part K_i of the partition.*

We now consider results with infinite d.

Theorem

1) (Galvin-Prikry; Mathias; Silver) For every partition $[\omega]^\omega = K_0 \dot\cup K_1$ with K_0 analytic there is $H \in [\omega]^\omega$ such that for some $i \in 2$,

$$[H]^\omega \subseteq K_i.$$

2) (Louveau-Shelah-Veličković) For every partition $K_0 \dot\cup K_1$ of the set of **strongly increasing sequences** *in 2^ω with K_0 analytic there is a perfect set P such that all strongly increasing sequences of P are in the same part K_i of the partition.*

Induced Ramsey theorems

Theorem

1) (Nešetřil-Rödl) For all finite ordered graphs A and B there is a finite ordered graph G such that for every partition $\binom{G}{A} = K_0 \dot\cup K_1$ of the set of induced copies of A in G there is an induced copy H of B in G such that $\binom{H}{A} \subseteq K_i$ for some i.

2) (Pouzet-Sauer; Sauer) Let R be the countable random graph. For every finite graph A there is a finite partition

$$\binom{R}{A} = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for every partition $\binom{R}{A} = K_0 \dot\cup K_1$ there is an induced copy H of R in R such that for all $i \in \{1, \ldots, n\}$, $\binom{H}{A} \cap C_i$ is included in one part of the partition $K_0 \dot\cup K_1$.

Induced Ramsey theorems

Theorem

1) (Nešetřil-Rödl) For all finite ordered graphs A and B there is a finite ordered graph G such that for every partition $\binom{G}{A} = K_0 \dot\cup K_1$ of the set of induced copies of A in G there is an induced copy H of B in G such that $\binom{H}{A} \subseteq K_i$ for some i.

2) (Pouzet-Sauer; Sauer) Let R be the countable random graph. For every finite graph A there is a finite partition

$$\binom{R}{A} = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for every partition $\binom{R}{A} = K_0 \dot\cup K_1$ there is an induced copy H of R in R such that for all $i \in \{1, \ldots, n\}$, $\binom{H}{A} \cap C_i$ is included in one part of the partition $K_0 \dot\cup K_1$.

Induced Ramsey theorems

Theorem

1) (Nešetřil-Rödl) For all finite ordered graphs A and B there is a finite ordered graph G such that for every partition $\binom{G}{A} = K_0 \dot\cup K_1$ of the set of induced copies of A in G there is an induced copy H of B in G such that $\binom{H}{A} \subseteq K_i$ for some i.

2) (Pouzet-Sauer; Sauer) Let R be the countable random graph. For every finite graph A there is a finite partition

$$\binom{R}{A} = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for every partition $\binom{R}{A} = K_0 \dot\cup K_1$ there is an induced copy H of R in R such that for all $i \in \{1, \ldots, n\}$, $\binom{H}{A} \cap C_i$ is included in one part of the partition $K_0 \dot\cup K_1$.

The $C_1, \ldots, C_n$ in Blass' theorem and in Sauer's theorem are called **basic partitions**. We call the individual C_i's **types**. The strongly increasing sequences in the Louveau-Shelah-Veličković theorem are one type of closed subsets of 2^ω.

We are after induced Ramsey results for graphs on 2^ω.

The $C_1, \ldots, C_n$ in Blass' theorem and in Sauer's theorem are called **basic partitions**. We call the individual C_i's **types**. The strongly increasing sequences in the Louveau-Shelah-Veličković theorem are one type of closed subsets of 2^ω.

We are after induced Ramsey results for graphs on 2^ω.

The $C_1, \ldots, C_n$ in Blass' theorem and in Sauer's theorem are called **basic partitions**. We call the individual C_i's **types**. The strongly increasing sequences in the Louveau-Shelah-Veličković theorem are one type of closed subsets of 2^ω.

We are after induced Ramsey results for graphs on 2^ω.

The $C_1, \ldots, C_n$ in Blass' theorem and in Sauer's theorem are called **basic partitions**. We call the individual C_i's **types**. The strongly increasing sequences in the Louveau-Shelah-Veličković theorem are one type of closed subsets of 2^ω.

We are after induced Ramsey results for graphs on 2^ω.

Clopen graphs

Definition

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, … if the **edge-relation** $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

A clopen graph $G = (V, E)$ is the same as a continuous coloring $c : [V]^2 \to 2$.

A natural question is this: Is there a clopen analog of the random graph on 2^ω?

Theorem

There is no universal clopen graph on 2^ω. The random graph R does not embed into any clopen graph on 2^ω.

Definition

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the **edge-relation** $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

A clopen graph $G = (V, E)$ is the same as a continuous coloring $c : [V]^2 \to 2$.

A natural question is this: Is there a clopen analog of the random graph on 2^ω?

Theorem

There is no universal clopen graph on 2^ω. The random graph R does not embed into any clopen graph on 2^ω.

Definition

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the **edge-relation** $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

A clopen graph $G = (V, E)$ is the same as a continuous coloring $c : [V]^2 \to 2$.

A natural question is this: Is there a clopen analog of the random graph on 2^ω?

Theorem

There is no universal clopen graph on 2^ω. The random graph R does not embed into any clopen graph on 2^ω.

Definition

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the **edge-relation** $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

A clopen graph $G = (V, E)$ is the same as a continuous coloring $c : [V]^2 \to 2$.

A natural question is this: Is there a clopen analog of the random graph on 2^ω?

Theorem

There is no universal clopen graph on 2^ω. The random graph R does not embed into any clopen graph on 2^ω.

Definition

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the **edge-relation** $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

A clopen graph $G = (V, E)$ is the same as a continuous coloring $c : [V]^2 \to 2$.

A natural question is this: Is there a clopen analog of the random graph on 2^ω?

Theorem

There is no universal clopen graph on 2^ω. The random graph R does not embed into any clopen graph on 2^ω.

Depth and the graph $G_{\min}$

Definition

Let $V \subseteq \omega^\omega$ be closed.

For distinct points $x, y \in V$ let
$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$.

A continuous coloring $c : [V]^2 \to 2$ and the corresponding graph on $(V, c^{-1}(1))$ are of **depth k** if for all $\{x, y\} \in V$, $c(x, y)$ only depends on

$$\{x, y\} \restriction (\Delta(x, y) + k).$$

Depth and the graph $G_{\min}$

Definition

Let $V \subseteq \omega^\omega$ be closed.

For distinct points $x, y \in V$ let
$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$.

A continuous coloring $c : [V]^2 \to 2$ and the corresponding graph on $(V, c^{-1}(1))$ are of **depth k** if for all $\{x, y\} \in V$, $c(x, y)$ only depends on

$$\{x, y\} \restriction (\Delta(x, y) + k).$$

Depth and the graph $G_{\min}$

Definition

Let $V \subseteq \omega^\omega$ be closed.

For distinct points $x, y \in V$ let
$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$.

A continuous coloring $c : [V]^2 \to 2$ and the corresponding graph on $(V, c^{-1}(1))$ are of **depth k** if for all $\{x, y\} \in V$, $c(x, y)$ only depends on

$$\{x, y\} \restriction (\Delta(x, y) + k).$$

Depth and the graph $G_{\min}$

Definition

Let $V \subseteq \omega^\omega$ be closed.

For distinct points $x, y \in V$ let
$\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$.

A continuous coloring $c : [V]^2 \to 2$ and the corresponding graph on $(V, c^{-1}(1))$ are of **depth k** if for all $\{x, y\} \in V$, $c(x, y)$ only depends on

$$\{x, y\} \restriction (\Delta(x, y) + k).$$

Example

Define a graph $G_{\min}$ on 2^ω by connecting two distinct vertices $x, y \in 2^\omega$ by an edge iff $\Delta(x, y)$ is odd. This graph is of depth 0.

Theorem (Sheu)

For every $d \in \omega \setminus \{0\}$ there is a finite partition

$$[2^\omega]^d = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for every continuous coloring $c : [2^\omega]^d \to 2$ there is a perfect set $P \subseteq 2^\omega$ supporting an induced copy of $G_{\min}$ in $G_{\min}$ such that for all $i \in \{1, \ldots, n\}$, c is constant on $P \cap C_i$.

Example

Define a graph G_{min} on 2^ω by connecting two distinct vertices $x, y \in 2^\omega$ by an edge iff $\Delta(x, y)$ is odd. This graph is of depth 0.

Theorem (Sheu)

For every $d \in \omega \setminus \{0\}$ there is a finite partition

$$[2^\omega]^d = C_1 \dot\cup \ldots \dot\cup C_n$$

such that for every continuous coloring $c : [2^\omega]^d \to 2$ there is a perfect set $P \subseteq 2^\omega$ supporting an induced copy of G_{min} in G_{min} such that for all $i \in \{1, \ldots, n\}$, c is constant on $P \cap C_i$.

Graphs of depth 1

Example

Let $V \subseteq \omega^\omega$ be compact. For each

$$t \in T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

choose a graph G_t on the set $\mathrm{succ}_{T(V)}(t)$ of immediate successors of t in $T(V)$.
Let two distinct vertices $x, y \in V$ form an edge in a graph G on V if $x \restriction (\Delta(x,y)+1)$ and $y \restriction (\Delta(x,y)+1)$ form an edge in G_t where $t = x \restriction \Delta(x,y)$. Then G is of depth 1.

Graphs of depth 1

Example

Let $V \subseteq \omega^\omega$ be compact. For each

$$t \in T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

choose a graph G_t on the set $\mathrm{succ}_{T(V)}(t)$ of immediate successors of t in $T(V)$.

Let two distinct vertices $x, y \in V$ form an edge in a graph G on V if $x \restriction (\Delta(x,y)+1)$ and $y \restriction (\Delta(x,y)+1)$ form an edge in G_t where $t = x \restriction \Delta(x,y)$. Then G is of depth 1.

Graphs of depth 1

Example

Let $V \subseteq \omega^\omega$ be compact. For each

$$t \in T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

choose a graph G_t on the set $\mathrm{succ}_{T(V)}(t)$ of immediate successors of t in $T(V)$.
Let two distinct vertices $x, y \in V$ form an edge in a graph G on V if $x \restriction (\Delta(x,y)+1)$ and $y \restriction (\Delta(x,y)+1)$ form an edge in G_t where $t = x \restriction \Delta(x,y)$. Then G is of depth 1.

Graphs of depth 1

Example

Let $V \subseteq \omega^\omega$ be compact. For each

$$t \in T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

choose a graph G_t on the set $\mathrm{succ}_{T(V)}(t)$ of immediate successors of t in $T(V)$.
Let two distinct vertices $x, y \in V$ form an edge in a graph G on V if $x \restriction (\Delta(x,y)+1)$ and $y \restriction (\Delta(x,y)+1)$ form an edge in G_t where $t = x \restriction \Delta(x,y)$. Then G is of depth 1.

Theorem (G., Goldstern, Kojman)

Every clopen graph on a Polish space can be decomposed into a small number of clopen graphs isomorphic to graphs of depth 1 on compact subsets of ω^ω.

Definition

Given a graph G on a topological space X, let $\mathrm{age}(G)$ be the class of finite graphs that embed into G and let $\mathrm{hage}(G)$ be the class of finite graphs that embed into every non-empty open subset of G.

Lemma

Let G be a graph of depth 1 on a compact subset of ω^ω *and let H be a clopen graph on a Polish space. If* $\mathrm{age}(G) \subseteq \mathrm{hage}(H)$, *then there is a continuous embedding of G into H.*

Theorem (G., Goldstern, Kojman)

Every clopen graph on a Polish space can be decomposed into a small number of clopen graphs isomorphic to graphs of depth 1 on compact subsets of ω^ω.

Definition

Given a graph G on a topological space X, let $\mathrm{age}(G)$ be the class of finite graphs that embed into G and let $\mathrm{hage}(G)$ be the class of finite graphs that embed into every non-empty open subset of G.

Lemma

Let G be a graph of depth 1 on a compact subset of ω^ω and let H be a clopen graph on a Polish space. If $\mathrm{age}(G) \subseteq \mathrm{hage}(H)$, then there is a continuous embedding of G into H.

Theorem (G., Goldstern, Kojman)

Every clopen graph on a Polish space can be decomposed into a small number of clopen graphs isomorphic to graphs of depth 1 on compact subsets of ω^ω.

Definition

Given a graph G on a topological space X, let $\text{age}(G)$ be the class of finite graphs that embed into G and let $\text{hage}(G)$ be the class of finite graphs that embed into every non-empty open subset of G.

Lemma

Let G be a graph of depth 1 on a compact subset of ω^ω and let H be a clopen graph on a Polish space. If $\text{age}(G) \subseteq \text{hage}(H)$*, then there is a continuous embedding of G into H.*

Corollary

There is a clopen graph $G_{\max}$ on a compact subset of ω^ω of depth 1 such that $\mathrm{hage}(G_{\max})$ is the class of all finite graphs. In particular, $G_{\max}$ universal for all clopen graphs of depth 1 on compact subsets of ω^ω. $G_{\max}$ is unique up to bi-embeddability.

More generally, to each class $\mathcal{C}$ of finite graphs we associate a graph $G_{\mathcal{C}}$ of depth 1 on a compact set $V_{\mathcal{C}} \subseteq \omega^\omega$ where for each $t \in T(V_{\mathcal{C}})$ the graph G_t on the immediate successors of t is in $\mathcal{C}$ and each graph in $\mathcal{C}$ appears cofinally often on each branch of $T(V_{\mathcal{C}})$.

$G_{\mathcal{C}}$ is unique up to bi-embeddability.

Corollary

There is a clopen graph $G_{\max}$ on a compact subset of ω^ω of depth 1 such that $\mathrm{hage}(G_{\max})$ is the class of all finite graphs. In particular, $G_{\max}$ universal for all clopen graphs of depth 1 on compact subsets of ω^ω. $G_{\max}$ is unique up to bi-embeddability.

More generally, to each class $\mathcal{C}$ of finite graphs we associate a graph $G_{\mathcal{C}}$ of depth 1 on a compact set $V_{\mathcal{C}} \subseteq \omega^\omega$ where for each $t \in T(V_{\mathcal{C}})$ the graph G_t on the immediate successors of t is in $\mathcal{C}$ and each graph in $\mathcal{C}$ appears cofinally often on each branch of $T(V_{\mathcal{C}})$.

$G_{\mathcal{C}}$ is unique up to bi-embeddability.

Corollary

There is a clopen graph $G_{\max}$ on a compact subset of ω^ω of depth 1 such that $\mathrm{hage}(G_{\max})$ is the class of all finite graphs. In particular, $G_{\max}$ universal for all clopen graphs of depth 1 on compact subsets of ω^ω. $G_{\max}$ is unique up to bi-embeddability.

More generally, to each class $\mathcal{C}$ of finite graphs we associate a graph $G_{\mathcal{C}}$ of depth 1 on a compact set $V_{\mathcal{C}} \subseteq \omega^\omega$ where for each $t \in T(V_{\mathcal{C}})$ the graph G_t on the immediate successors of t is in $\mathcal{C}$ and each graph in $\mathcal{C}$ appears cofinally often on each branch of $T(V_{\mathcal{C}})$.

$G_{\mathcal{C}}$ is unique up to bi-embeddability.

Order

We will use the structural Ramsey theorem by Nešetřil and Rödl, which talks about finite ordered graphs.

With a $\mathcal{C}$ is a class of finite ordered graphs we associate a graph $G_{\mathcal{C}}$ on a compact subset $V_{\mathcal{C}}$ of ω^ω in the same way as before, this times considering the graphs G_t as ordered graphs with respect to the lexicographic ordering on the set $\mathrm{succ}_{T(V_{\mathcal{C}})}(t)$.

Order

We will use the structural Ramsey theorem by Nešetřil and Rödl, which talks about finite ordered graphs.

With a $\mathcal{C}$ is a class of finite ordered graphs we associate a graph $G_{\mathcal{C}}$ on a compact subset $V_{\mathcal{C}}$ of ω^ω in the same way as before, this times considering the graphs G_t as ordered graphs with respect to the lexicographic ordering on the set $\mathrm{succ}_{T(V_{\mathcal{C}})}(t)$.

Order

We will use the structural Ramsey theorem by Nešetřil and Rödl, which talks about finite ordered graphs.

With a $\mathcal{C}$ is a class of finite ordered graphs we associate a graph $G_{\mathcal{C}}$ on a compact subset $V_{\mathcal{C}}$ of ω^ω in the same way as before, this times considering the graphs G_t as ordered graphs with respect to the lexicographic ordering on the set $\mathrm{succ}_{T(V_{\mathcal{C}})}(t)$.

Partitions of closed sets

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed.
A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed.
A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed.
A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed. A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed.
A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

Let $\mathcal{C}$ be a class of finite ordered graphs and $A, B \subseteq V_{\mathcal{C}}$ closed. A map $f : A \to B$ is a **strong isomorphism** if the following hold:

1. f is an isomorphism between the induced subgraphs of $G_{\mathcal{C}}$ on the sets A and B.
2. f is increasing with respect to the lexicographic ordering on ω^ω.
3. If $x_0, x_1, x_2, x_3 \in A$ are such that $x_0 \neq x_1$ and $x_2 \neq x_3$, then

$$\Delta(x_0, x_1) \leq \Delta(x_2, x_3) \Rightarrow \Delta(f(x_0), f(x_1)) \leq \Delta(f(x_2), f(x_3)).$$

Note that (3) implies that if $x_1, \ldots, x_n \in A$ all split from each other at the same level, then so do $f(x_1), \ldots, f(x_n)$.

Definition

The **type** of a closed set $A \subseteq V_{\mathcal{C}}$ is its strong isomorphism type.

A type τ is **fully splitting** if for all closed sets $A \subseteq V_{\mathcal{C}}$ and all $t \in T(A)$ either $\mathrm{succ}_{T(A)}(t)$ is a singleton or equals $\mathrm{succ}_{T_{\mathcal{C}}}(t)$.

A class $\mathcal{C}$ of finite ordered graphs is **Ramsey** if the Nešetřil-Rödl theorem holds for graphs in $\mathcal{C}$.

A class $\mathcal{C}$ of finite ordered graphs has the **joint embedding property** if any two elements A and B of $\mathcal{C}$ embed into a third $C \in \mathcal{C}$.

Definition

The **type** of a closed set $A \subseteq V_\mathcal{C}$ is its strong isomorphism type.

A type τ is **fully splitting** if for all closed sets $A \subseteq V_\mathcal{C}$ and all $t \in T(A)$ either $\mathrm{succ}_{T(A)}(t)$ is a singleton or equals $\mathrm{succ}_{T_\mathcal{C}}(t)$.

A class $\mathcal{C}$ of finite ordered graphs is **Ramsey** if the Nešetřil-Rödl theorem holds for graphs in $\mathcal{C}$.

A class $\mathcal{C}$ of finite ordered graphs has the **joint embedding property** if any two elements A and B of $\mathcal{C}$ embed into a third $C \in \mathcal{C}$.

Definition

The **type** of a closed set $A \subseteq V_\mathcal{C}$ is its strong isomorphism type.

A type τ is **fully splitting** if for all closed sets $A \subseteq V_\mathcal{C}$ and all $t \in T(A)$ either $\mathrm{succ}_{T(A)}(t)$ is a singleton or equals $\mathrm{succ}_{T_\mathcal{C}}(t)$.

A class $\mathcal{C}$ of finite ordered graphs is **Ramsey** if the Nešetřil-Rödl theorem holds for graphs in $\mathcal{C}$.

A class $\mathcal{C}$ of finite ordered graphs has the **joint embedding property** if any two elements A and B of $\mathcal{C}$ embed into a third $C \in \mathcal{C}$.

Definition

The **type** of a closed set $A \subseteq V_{\mathcal{C}}$ is its strong isomorphism type.

A type τ is **fully splitting** if for all closed sets $A \subseteq V_{\mathcal{C}}$ and all $t \in T(A)$ either $\mathrm{succ}_{T(A)}(t)$ is a singleton or equals $\mathrm{succ}_{T_{\mathcal{C}}}(t)$.

A class $\mathcal{C}$ of finite ordered graphs is **Ramsey** if the Nešetřil-Rödl theorem holds for graphs in $\mathcal{C}$.

A class $\mathcal{C}$ of finite ordered graphs has the **joint embedding property** if any two elements A and B of $\mathcal{C}$ embed into a third $C \in \mathcal{C}$.

Theorem

Let $\mathcal{C}$ be a class of finite ordered graphs.

Let τ be the type of a closed subset of $V_{\mathcal{C}}$.

Assume that $\mathcal{C}$ is Ramsey with the joint embedding property or that τ is fully splitting.

Then for every partition $K_0 \dot{\cup} K_1$ of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic, there is a compact set $P \subseteq V_{\mathcal{C}}$ supporting an induced copy of $G_{\mathcal{C}}$ such that all closed subsets of P of type τ are in the same part of the partition.

This gives an induced Ramsey theorem for $G_{\max}$ and generalizes Sheu's result.

Theorem

Let $\mathcal{C}$ be a class of finite ordered graphs.

Let τ be the type of a closed subset of $V_{\mathcal{C}}$.

Assume that $\mathcal{C}$ is Ramsey with the joint embedding property or that τ is fully splitting.

Then for every partition $K_0 \dot\cup K_1$ of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic, there is a compact set $P \subseteq V_{\mathcal{C}}$ supporting an induced copy of $G_{\mathcal{C}}$ such that all closed subsets of P of type τ are in the same part of the partition.

This gives an induced Ramsey theorem for $G_{\max}$ and generalizes Sheu's result.

Theorem

Let $\mathcal{C}$ be a class of finite ordered graphs.

Let τ be the type of a closed subset of $V_{\mathcal{C}}$.

Assume that $\mathcal{C}$ is Ramsey with the joint embedding property or that τ is fully splitting.

Then for every partition $K_0 \dot\cup K_1$ of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic, there is a compact set $P \subseteq V_{\mathcal{C}}$ supporting an induced copy of $G_{\mathcal{C}}$ such that all closed subsets of P of type τ are in the same part of the partition.

This gives an induced Ramsey theorem for $G_{\max}$ and generalizes Sheu's result.

Theorem

Let $\mathcal{C}$ be a class of finite ordered graphs.

Let τ be the type of a closed subset of $V_{\mathcal{C}}$.

Assume that $\mathcal{C}$ is Ramsey with the joint embedding property or that τ is fully splitting.

Then for every partition $K_0 \dot{\cup} K_1$ of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic, there is a compact set $P \subseteq V_{\mathcal{C}}$ supporting an induced copy of $G_{\mathcal{C}}$ such that all closed subsets of P of type τ are in the same part of the partition.

This gives an induced Ramsey theorem for $G_{\max}$ and generalizes Sheu's result.

Theorem

Let $\mathcal{C}$ be a class of finite ordered graphs.

Let τ be the type of a closed subset of $V_{\mathcal{C}}$.

Assume that $\mathcal{C}$ is Ramsey with the joint embedding property or that τ is fully splitting.

Then for every partition $K_0 \dot{\cup} K_1$ of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic, there is a compact set $P \subseteq V_{\mathcal{C}}$ supporting an induced copy of $G_{\mathcal{C}}$ such that all closed subsets of P of type τ are in the same part of the partition.

This gives an induced Ramsey theorem for $G_{\max}$ and generalizes Sheu's result.

Sketch of the proof

We follow Louveau-Shelah-Veličković.

We define two forcing notions $\mathbb{P}$ and $\mathbb{P}_\tau$, $\mathbb{P}_\tau$ adding a generic closed subset of $V_\mathcal{C}$ of type τ and $\mathbb{P}$ adding a closed subset P of $V_\mathcal{C}$ supporting a copy of $G_\mathcal{C}$.

The conditions of $\mathbb{P}$ are pairs (S, n) where S is a subtree of $T(V_\mathcal{C})$ whose set $[S]$ of branches supports a copy of $G_\mathcal{C}$.

A condition $(S, n) \in \mathbb{P}$ extends a condition $(T, m) \in \mathbb{P}$ if $S \subseteq T$, $m \leq n$, and $S \cap \omega^{\leq m} = T \cap \omega^{\leq m}$.

Sketch of the proof

We follow Louveau-Shelah-Veličković.

We define two forcing notions $\mathbb{P}$ and $\mathbb{P}_\tau$, $\mathbb{P}_\tau$ adding a generic closed subset of $V_\mathcal{C}$ of type τ and $\mathbb{P}$ adding a closed subset P of $V_\mathcal{C}$ supporting a copy of $G_\mathcal{C}$.

The conditions of $\mathbb{P}$ are pairs (S, n) where S is a subtree of $T(V_\mathcal{C})$ whose set $[S]$ of branches supports a copy of $G_\mathcal{C}$.

A condition $(S, n) \in \mathbb{P}$ extends a condition $(T, m) \in \mathbb{P}$ if $S \subseteq T$, $m \leq n$, and $S \cap \omega^{\leq m} = T \cap \omega^{\leq m}$.

Sketch of the proof

We follow Louveau-Shelah-Veličković.

We define two forcing notions $\mathbb{P}$ and $\mathbb{P}_\tau$, $\mathbb{P}_\tau$ adding a generic closed subset of $V_{\mathcal{C}}$ of type τ and $\mathbb{P}$ adding a closed subset P of $V_{\mathcal{C}}$ supporting a copy of $G_{\mathcal{C}}$.

The conditions of $\mathbb{P}$ are pairs (S, n) where S is a subtree of $T(V_{\mathcal{C}})$ whose set $[S]$ of branches supports a copy of $G_{\mathcal{C}}$.

A condition $(S, n) \in \mathbb{P}$ extends a condition $(T, m) \in \mathbb{P}$ if $S \subseteq T$, $m \leq n$, and $S \cap \omega^{\leq m} = T \cap \omega^{\leq m}$.

Sketch of the proof

We follow Louveau-Shelah-Veličković.

We define two forcing notions $\mathbb{P}$ and $\mathbb{P}_\tau$, $\mathbb{P}_\tau$ adding a generic closed subset of $V_{\mathcal{C}}$ of type τ and $\mathbb{P}$ adding a closed subset P of $V_{\mathcal{C}}$ supporting a copy of $G_{\mathcal{C}}$.

The conditions of $\mathbb{P}$ are pairs (S, n) where S is a subtree of $T(V_{\mathcal{C}})$ whose set $[S]$ of branches supports a copy of $G_{\mathcal{C}}$.

A condition $(S, n) \in \mathbb{P}$ extends a condition $(T, m) \in \mathbb{P}$ if $S \subseteq T$, $m \leq n$, and $S \cap \omega^{\leq m} = T \cap \omega^{\leq m}$.

Sketch of the proof

We follow Louveau-Shelah-Veličković.

We define two forcing notions $\mathbb{P}$ and $\mathbb{P}_\tau$, $\mathbb{P}_\tau$ adding a generic closed subset of $V_\mathcal{C}$ of type τ and $\mathbb{P}$ adding a closed subset P of $V_\mathcal{C}$ supporting a copy of $G_\mathcal{C}$.

The conditions of $\mathbb{P}$ are pairs (S, n) where S is a subtree of $T(V_\mathcal{C})$ whose set $[S]$ of branches supports a copy of $G_\mathcal{C}$.

A condition $(S, n) \in \mathbb{P}$ extends a condition $(T, m) \in \mathbb{P}$ if $S \subseteq T$, $m \leq n$, and $S \cap \omega^{\leq m} = T \cap \omega^{\leq m}$.

$\mathbb{P}_\tau$ is the suborder of $\mathbb{P}$ consisting of conditions (S, n) with the additional property that there is a closed set $A \subseteq [S]$ of type τ with $T(A) \cap \omega^{\leq n} = S \cap \omega^{\leq n}$.

For conditions $(S, n), (T, m) \in \mathbb{P}_\tau$, (S, n) is a **pure extension** of (T, m) if (S, n) extends (T, m) and $n = m$.

Lemma (Pure extension property of $\mathbb{P}_\tau$)

Let φ be a statement in the forcing language of $\mathbb{P}_\tau$. Then every condition $(S, n) \in \mathbb{P}_\tau$ has a pure extension deciding φ.

The proof of this lemma uses the Halpern-Läuchli theorem and the properties of $\mathcal{C}$ and τ.

$\mathbb{P}_\tau$ is the suborder of $\mathbb{P}$ consisting of conditions (S, n) with the additional property that there is a closed set $A \subseteq [S]$ of type τ with $T(A) \cap \omega^{\leq n} = S \cap \omega^{\leq n}$.

For conditions $(S, n), (T, m) \in \mathbb{P}_\tau$, (S, n) is a **pure extension** of (T, m) if (S, n) extends (T, m) and $n = m$.

Lemma (Pure extension property of $\mathbb{P}_\tau$)

Let φ be a statement in the forcing language of $\mathbb{P}_\tau$. Then every condition $(S, n) \in \mathbb{P}_\tau$ has a pure extension deciding φ.

The proof of this lemma uses the Halpern-Läuchli theorem and the properties of $\mathcal{C}$ and τ.

$\mathbb{P}_\tau$ is the suborder of $\mathbb{P}$ consisting of conditions (S, n) with the additional property that there is a closed set $A \subseteq [S]$ of type τ with $T(A) \cap \omega^{\leq n} = S \cap \omega^{\leq n}$.

For conditions $(S, n), (T, m) \in \mathbb{P}_\tau$, (S, n) is a **pure extension** of (T, m) if (S, n) extends (T, m) and $n = m$.

Lemma (Pure extension property of $\mathbb{P}_\tau$)

Let φ be a statement in the forcing language of $\mathbb{P}_\tau$. Then every condition $(S, n) \in \mathbb{P}_\tau$ has a pure extension deciding φ.

The proof of this lemma uses the Halpern-Läuchli theorem and the properties of $\mathcal{C}$ and τ.

$\mathbb{P}_\tau$ is the suborder of $\mathbb{P}$ consisting of conditions (S, n) with the additional property that there is a closed set $A \subseteq [S]$ of type τ with $T(A) \cap \omega^{\leq n} = S \cap \omega^{\leq n}$.

For conditions $(S, n), (T, m) \in \mathbb{P}_\tau$, (S, n) is a **pure extension** of (T, m) if (S, n) extends (T, m) and $n = m$.

Lemma (Pure extension property of $\mathbb{P}_\tau$)

Let φ be a statement in the forcing language of $\mathbb{P}_\tau$. Then every condition $(S, n) \in \mathbb{P}_\tau$ has a pure extension deciding φ.

The proof of this lemma uses the Halpern-Läuchli theorem and the properties of $\mathcal{C}$ and τ.

Lemma

If P is the generic closed set added by forcing with $\mathbb{P}$, then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic.

Now let $K_0 \dot\cup K_1$ be a partition of the closed subsets of $V_\mathcal{C}$ of type τ with K_0 analytic and let M be a countable transitive model of enough set theory containing the relevant data.

Recall that by Mostowski absoluteness, analytic properties are absolute over M.

By the pure extension property of $\mathbb{P}_\tau$, we find a condition $(S, 0) \in \mathbb{P}_\tau$ that decides the color of the generic closed set of type τ added by $\mathbb{P}_\tau$.

Lemma

If P is the generic closed set added by forcing with $\mathbb{P}$, then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic.

Now let $K_0 \dot\cup K_1$ be a partition of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic and let M be a countable transitive model of enough set theory containing the relevant data.

Recall that by Mostowski absoluteness, analytic properties are absolute over M.

By the pure extension property of $\mathbb{P}_\tau$, we find a condition $(S, 0) \in \mathbb{P}_\tau$ that decides the color of the generic closed set of type τ added by $\mathbb{P}_\tau$.

Lemma

If P is the generic closed set added by forcing with $\mathbb{P}$, then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic.

Now let $K_0 \dot{\cup} K_1$ be a partition of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic and let M be a countable transitive model of enough set theory containing the relevant data.

Recall that by Mostowski absoluteness, analytic properties are absolute over M.

By the pure extension property of $\mathbb{P}_\tau$, we find a condition $(S, 0) \in \mathbb{P}_\tau$ that decides the color of the generic closed set of type τ added by $\mathbb{P}_\tau$.

Lemma

If P is the generic closed set added by forcing with $\mathbb{P}$, then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic.

Now let $K_0 \dot{\cup} K_1$ be a partition of the closed subsets of $V_{\mathcal{C}}$ of type τ with K_0 analytic and let M be a countable transitive model of enough set theory containing the relevant data.

Recall that by Mostowski absoluteness, analytic properties are absolute over M.

By the pure extension property of $\mathbb{P}_\tau$, we find a condition $(S, 0) \in \mathbb{P}_\tau$ that decides the color of the generic closed set of type τ added by $\mathbb{P}_\tau$.

Now we re-interpret $(S, 0)$ as a condition in $\mathbb{P}$.

Let P be the closed set corresponding to an $\mathbb{P}$-generic filter over M that contains $(S, 0)$.

Then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic over M and comes from a filter containing $(S, 0)$.

In particular, all closed subsets of P of type τ have the color that was decided by $(S, 0)$.

It follows that P works for the theorem. □

Now we re-interpret $(S, 0)$ as a condition in $\mathbb{P}$.

Let P be the closed set corresponding to an $\mathbb{P}$-generic filter over M that contains $(S, 0)$.

Then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic over M and comes from a filter containing $(S, 0)$.

In particular, all closed subsets of P of type τ have the color that was decided by $(S, 0)$.

It follows that P works for the theorem. $\square$

Now we re-interpret $(S, 0)$ as a condition in $\mathbb{P}$.

Let P be the closed set corresponding to an $\mathbb{P}$-generic filter over M that contains $(S, 0)$.

Then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic over M and comes from a filter containing $(S, 0)$.

In particular, all closed subsets of P of type τ have the color that was decided by $(S, 0)$.

It follows that P works for the theorem. □

Now we re-interpret $(S, 0)$ as a condition in $\mathbb{P}$.

Let P be the closed set corresponding to an $\mathbb{P}$-generic filter over M that contains $(S, 0)$.

Then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic over M and comes from a filter containing $(S, 0)$.

In particular, all closed subsets of P of type τ have the color that was decided by $(S, 0)$.

It follows that P works for the theorem. □

Now we re-interpret $(S, 0)$ as a condition in $\mathbb{P}$.

Let P be the closed set corresponding to an $\mathbb{P}$-generic filter over M that contains $(S, 0)$.

Then every closed subset of P of type τ is $\mathbb{P}_\tau$-generic over M and comes from a filter containing $(S, 0)$.

In particular, all closed subsets of P of type τ have the color that was decided by $(S, 0)$.

It follows that P works for the theorem. $\square$

Thank you!