A theory of non-special trees, and a generalization of the Balanced Baumgartner-Hajnal-Todorčević Theorem

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Special trees arose initially in the context of Souslin's problem.

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Is every ccc dense linear ordering necessarily separable?

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Theorem (Kupera, 1935)

$$\exists \text{ Souslin line} \iff \exists \text{ Souslin tree.}$$

Definition

A tree T is **Souslin** if:

- it has height ω_1,
- every chain is countable, and
- every antichain is countable.

We now know that Souslin's problem is independent of ZFC. Among other constructions, we have:

$$\diamondsuit \implies \exists \text{ Souslin tree}$$
$$\text{MA}_{\aleph_1} \implies \nexists \text{ Souslin tree}$$

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There are several constructions giving:

Theorem

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$$f : T \to \mathbb{Q}.$$

Equivalently, T is special if we can write it as a union of countably many antichains.

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It is clear that a special Aronszajn tree cannot be Souslin.

It turns out that using $\mathrm{MA}_{\aleph_1}$, not only are there no Souslin trees, but:

Theorem (Baumgartner, Malitz & Reinhardt, 1970)

$$MA_{\aleph_1} \implies \textit{every Aronszajn tree is special.}$$

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This gives the impression that non-special trees are pathological. However, this is only because until now we have restricted our attention to Aronszajn trees.

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Being special or non-special is in some sense a condition on the height of the tree, the number of antichains.
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Main motivational point: A non-special tree is in some sense a generalization of the ordinal ω_1, since ω_1 is the simplest non-special tree.
So the goal is to determine what facts about ω_1 are true for non-special trees as well.

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So the goal is to determine what facts about ω_1 are true for non-special trees as well.
Another example of a non-special tree is $\sigma\mathbb{Q}$, defined (by Kurepa) to be the collection of well-ordered sequences of rationals, ordered by end-extension.

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We will explain how many standard concepts that are defined on ordinals, such as regressive functions, diagonal unions, normal ideals, and stationary and nonstationary subsets can be generalized to non-special trees.

Always assume T is a tree with order relation $<_T$.

Definition

For any tree T and node $t \in T$, we define:

$$\text{Predecessors of } t: \qquad t{\downarrow} = \{s \in T : s <_T t\}$$

$$\text{Cone above } t: \qquad t{\uparrow} = \begin{cases} \{s \in T : t <_T s\} & \text{if } t \neq \emptyset \\ T & \text{if } t = \emptyset. \end{cases}$$

When discussing diagonal unions, it will be crucial that $t{\uparrow}$ be defined so as not to include t. However, it will be convenient to make an exception for the cone above the root node $\emptyset$, to allow the root to be in the "cone above" some node.

Definition

Let T be a tree. For a collection

$$\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T),$$

we define its diagonal union to be

$$\mathop{\nabla}_{t \in T} A_t = \bigcup_{t \in T} \left(A_t \cap t\uparrow \right).$$

This generalizes the definition for subsets of a cardinal.

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This generalizes the definition for subsets of a cardinal.
Basic intuition: When taking the diagonal union of sets A_t, the only part of each A_t that contributes to the result is $A_t \cap t\uparrow$.

Lemma

For any tree T and any collection $\langle A_t \rangle_{t \in T} \subseteq \mathcal{P}(T)$, we have:

$$\nabla_{t \in T} A_t = \left\{ s \in T : s \in A_\emptyset \cup \bigcup_{t <_T s} A_t \right\} .$$

Definition

Let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal. We define

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Some easy facts about $\nabla \mathcal{I}$:

Lemma

If $\mathcal{I}$ is any ideal on T, then:

- $\mathcal{I} \subseteq \nabla \mathcal{I}$.
- *$\nabla \mathcal{I}$ is also an ideal (though not necessarily proper).*
- *For any cardinal λ, if $\mathcal{I}$ is λ-complete, then so is $\nabla \mathcal{I}$.*

Notice that the statement $\mathcal{I} \subseteq \nabla \mathcal{I}$ of the Lemma relies crucially on our earlier convention that $\emptyset \in \emptyset\uparrow$. Otherwise any set containing the root would never be in $\nabla \mathcal{I}$.

Definition

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Lemma

Let T be a tree, and let $\mathcal{I} \subseteq \mathcal{P}(T)$ be an ideal on T. Then

$$\nabla \mathcal{I} = \left\{ X \subseteq T : \begin{array}{c} \exists \text{ regressive } f : X \to T \\ (\forall t \in T)\left[f^{-1}(t) \in \mathcal{I} \right] \end{array} \right\}.$$

Corollary

Taking complements, a set X is $(\nabla \mathcal{I})$-positive iff every regressive function on X is constant on an $\mathcal{I}^+$-set.

Corollary

For any ideal $\mathcal{I} \subseteq \mathcal{P}(T)$, the following are equivalent:

1. *$\mathcal{I}$ is closed under diagonal unions, that is, $\nabla \mathcal{I} = \mathcal{I}$;*
2. *If $X \in \mathcal{I}^+$, and $f : X \to T$ is a regressive function, then f must be constant on some $\mathcal{I}^+$-set, that is, $(\exists t \in T) f^{-1}(t) \in \mathcal{I}^+$.*

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Definition

An ideal $\mathcal{I}$ on T is normal if it is closed under diagonal unions (that is, $\nabla\mathcal{I} = \mathcal{I}$), or equivalently, if every regressive function on an $\mathcal{I}^+$ set must be constant on an $\mathcal{I}^+$ set.

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For any ideal $\mathcal{I} \subseteq \mathcal{P}(T)$*, the following are equivalent:*

1. $\mathcal{I}$ *is closed under diagonal unions, that is,* $\nabla \mathcal{I} = \mathcal{I}$*;*
2. *If* $X \in \mathcal{I}^+$*, and* $f : X \to T$ *is a regressive function, then* f *must be constant on some* $\mathcal{I}^+$*-set, that is,* $(\exists t \in T) f^{-1}(t) \in \mathcal{I}^+$*.*

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A natural question arises: For a given ideal, how many times must we iterate the diagonal union operation ∇ before the operation stabilizes and we obtain a normal ideal? In particular, when is ∇ idempotent? The following lemma gives us a substantial class of ideals for which the answer is *one*, and this will be a useful tool in later proofs:

Lemma (Idempotence Lemma)

Let $\lambda = \mathrm{ht}(T)$*, and suppose* λ *is any cardinal. If* $\mathcal{I}$ *is a* λ*-complete ideal on* T*, then* $\nabla \nabla \mathcal{I} = \nabla \mathcal{I}$*, that is,* $\nabla \mathcal{I}$ *is normal.*

PROOF:

Let $X \in \nabla \nabla \mathcal{I}$. We must show $X \in \nabla \mathcal{I}$.

As $X \in \nabla \nabla \mathcal{I}$, we can write

$$X = \bigtriangledown_{t \in T} A_t,$$

where each $A_t \in \nabla \mathcal{I}$. For each $t \in T$, we can write

$$A_t = \bigtriangledown_{s \in T} B_t^s,$$

where each $B_t^s \in \mathcal{I}$.

Notice that for each $t \in T$, the only part of A_t that contributes to X is the part within $t\uparrow$. For each $s, t \in T$, the only part of B_t^s that contributes to A_t is the part within $s\uparrow$. We therefore have:

- If s and t are incomparable in T, we have $s\uparrow \cap t\uparrow = \emptyset$, so B_t^s does not contribute anything to X;
- If $t \leq_T s$ then $s\uparrow \cap t\uparrow = s\uparrow$, so the only part of B_t^s that contributes to X is within $s\uparrow$;
- If $s \leq_T t$ then $s\uparrow \cap t\uparrow = t\uparrow$, so the only part of B_t^s that contributes to X is within $t\uparrow$.

We collect the sets B_t^s whose contribution to X lies within any $r\uparrow$. We define, for each $r \in T$,

$$D_r = \bigcup_{t \leq_T r} B_t^r \cup \bigcup_{s \leq_T r} B_r^s.$$

Since $\mathcal{I}$ is λ-complete and each r has height $< \lambda$, we have $D_r \in \mathcal{I}$.

Claim

We have

$$X = \bigvee_{r \in T} D_r .$$

It follows that $X \in \bigvee \mathcal{I}$, as required. □

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As an analogue to the ideal of bounded sets in κ^+, we consider the collection of κ-special subtrees of T:

Definition

Let T be a tree of height κ^+. We say that $U \subseteq T$ is a κ-special subtree of T if U can be written as a union of $\leq \kappa$ many antichains. That is, U is a κ-special subtree of T if

$$U = \bigcup_{\alpha<\kappa} A_\alpha,$$

where each $A_\alpha \subseteq T$ is an antichain, or equivalently, if

$$\exists f : U \to \kappa \, (\forall t, u \in U) \, [t <_T u \implies f(t) \neq f(u)] \, .$$

The collection of κ-special subtrees of T is clearly a κ^+-complete ideal on T, and it is proper iff T is itself non-κ-special.

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The cardinal κ^+ itself is an example of a non-κ-special tree of height κ^+. Letting $T = \kappa^+$, we see that the κ-special subtrees of κ^+ are precisely the bounded subsets of κ^+, supporting the choice of analogue.

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The problem is that we cannot reasonably define a club subset of a tree in a way that is analogous to a club subset of a cardinal. Instead, we recall the alternate characterization of stationary and nonstationary subsets given by Neumer:

Theorem (Neumer, 1951)

For a regular uncountable cardinal λ, and a set $X \subseteq \lambda$, the following are equivalent:

- *X intersects every club set of λ;*
- *For every regressive function $f : X \to \lambda$, there is some $\alpha < \lambda$ such that $f^{-1}(\alpha)$ is unbounded below λ. (In our terminology: $X \notin \nabla \mathcal{I}$, where $\mathcal{I}$ is the ideal of bounded sets.)*

We use this characterization to motivate similar definitions on trees.

Definition

Let $B \subseteq T$, where T is a tree of height κ^+. We say that B is a nonstationary subtree of T if we can write

$$B = \nabla_{t \in T} A_t,$$

where each A_t is a κ-special subtree of T. We may, for emphasis, refer to B as κ-nonstationary. If B cannot be written this way, then B is a stationary subtree of T.

We define NS_κ^T to be the collection of nonstationary subtrees of T. That is, NS_κ^T is the diagonal union of the ideal of κ-special subtrees of T. (The subscript κ is for emphasis and may sometimes be omitted.)

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In the case that $T = \kappa^+$, Neumer's Theorem tells us that NS_κ^T is identical to the collection of sets whose complements include a club subset of κ^+, so the analogy is correct.

Our definitions here are new, and in particular are different from Stevo's earlier use of NS_T. Stevo defines NS_T as an ideal on the cardinal κ^+, consisting of subsets of κ^+ that are said to be nonstationary in or with respect to T, while we define NS_κ^T as an ideal on the tree T itself, consisting of sets that are nonstationary subsets of T.

Our definitions will allow greater flexibility in stating and proving the relevant results. In particular, we can discuss the membership of arbitrary subsets of the tree in the ideal NS_κ^T, rather than only those of the form $T \upharpoonright X$ for some $X \subseteq \kappa^+$.

Some easy facts about NS_κ^T:

Lemma

Fix a tree T of height κ^+. Then:

- *Every κ-special subtree of T is a nonstationary subtree.*
- *Furthermore, NS_κ^T is a κ^+-complete ideal on T.*

The converse of the first conclusion of this Lemma is false: In the special case where T is just the cardinal κ^+, there exist unbounded nonstationary subsets of κ^+, so any such set is a nonstationary subtree of κ^+ that is not κ-special. This also means that the ideal of bounded subsets of κ^+ is not normal, so that in general the ideal of κ-special subtrees of a tree T is not a normal ideal.

However, we do have:

Theorem

For any tree T of height κ^+, the ideal NS_κ^T is a normal ideal on T.

Proof.

This follows from the Idempotence Lemma, since the ideal of κ-special subtrees is κ^+-complete. □

This theorem tells us that $\nabla NS_\kappa^T = NS_\kappa^T$. Equivalently: If B is a stationary subtree of T, meaning that every regressive function on B is constant on a non-κ-special subtree of T, then in fact every regressive function on B is constant on a stationary subtree of T. So for any tree T of height κ^+, the main tool for extracting subtrees using regressive functions should be the ideal NS_κ^T, rather than the ideal of κ-special subtrees of T.

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The ideal NS_κ^T will be useful if we know that it is proper. When can we guarantee that $T \notin NS_\kappa^T$?
Obviously, if a tree is special, then all of its subtrees are special and therefore nonstationary. The following theorem gives the converse, establishing the significance of using a nonspecial tree as our ambient space:

Theorem (Pressing-Down Lemma for Trees: Todorčević, 1981)

Suppose T is a non-κ-special tree. Then NS_κ^T is a proper ideal on T, that is, $T \notin NS_\kappa^T$.

The Pressing-Down Lemma for Trees is a generalization to non-special trees of a theorem of Dushnik (1931) on successor cardinals, which itself was a generalization of Alexandroff and Urysohn's theorem (1929) on ω_1.

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The following facts are straightforward:

Lemma

Let T be any tree of height κ^+, and let $X, C \subseteq \kappa^+$. Then:

1. *If $|X| \leq \kappa$ then $T \restriction X$ is a κ-special subtree of T.*
2. *If X is a nonstationary subset of κ^+, then $T \restriction X \in NS_\kappa^T$.*
3. *In particular, the set of successor nodes of T is a nonstationary subtree of T.*
4. *If C is a club subset of κ^+, then $T \restriction C \in (NS_\kappa^T)^*$.*
5. *If T is a non-κ-special tree and C is a club subset of κ^+, then $T \restriction C \notin NS_\kappa^T$.*

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A partial analogue to this theorem for trees is:

Theorem (Todorčević, 1985)

If T is a non-κ-special tree, then the subtree

$$T \restriction S^{\kappa^+}_{\mathrm{cf}(\kappa)} = \{t \in T : \mathrm{cf}(\mathrm{ht}(t)) = \mathrm{cf}(\kappa)\}$$

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Of course, in the case where T has height ω_1 (that is, where $\kappa = \omega$), this theorem provides no new information, because the set of ordinals with countable cofinality is just the set of limit ordinals below ω_1 and is therefore a club subset. But when $\kappa > \omega$, it provides a nontrivial example of a stationary subtree of T whose complement is not (necessarily) nonstationary.

Theorem (Main Theorem)

Let ν and κ be infinite cardinals such that $\nu^{<\kappa} = \nu$. Then for any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have

$$\text{non-}\nu\text{-special tree} \rightarrow (\kappa + \xi)^2_k \, .$$

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The arrow notation means:
For any non-ν-special tree T, and any colouring $c : [T]^2 \rightarrow k$, there is a chain $X \subseteq T$ of order type $\kappa + \xi$ that is i-homogeneous, that is, $c''[X]^2 = \{i\}$ for some colour $i < k$.

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It turns out that

$$\mathrm{cf}\left(2^{<\kappa}\right) \geq \kappa \iff \left(2^{<\kappa}\right)^{<\kappa} = 2^{<\kappa},$$

so that we can set $\nu = 2^{<\kappa}$ in the Main Theorem precisely iff $\mathrm{cf}(2^{<\kappa}) \geq \kappa$.
The Main Theorem then becomes:

Corollary

Let κ be any infinite cardinal satisfying $\mathrm{cf}(2^{<\kappa}) \geq \kappa$. Then for any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have

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$$\textit{non-}(2^{<\kappa})\textit{-special tree} \rightarrow (\kappa + \xi)^2_k .$$

In particular, any regular cardinal κ always satisfies $\mathrm{cf}(2^{<\kappa}) \geq \kappa$.

Of course the simplest example of a non-$(2^{<\kappa})$-special tree is the cardinal $(2^{<\kappa})^+$, and in this special case we have:

Corollary (Balanced Baumgartner-Hajnal-Todorčević Theorem, 1991)

Let κ be any regular cardinal. Then for any ordinal ξ such that $2^{|\xi|} < \kappa$, and any natural number k, we have

$$\left(2^{<\kappa}\right)^+ \to (\kappa + \xi)^2_k .$$

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This in turn was a partial strengthening of:

Theorem (Erdős-Rado, 1956)

For any infinite cardinal κ and any cardinal $\gamma < \mathrm{cf}(\kappa)$,

$$\left(2^{<\kappa}\right)^+ \to (\kappa + 1)^2_\gamma \,.$$

(Greater ordinal result at the cost of fewer colours.)

Examples:
Set $\kappa = \aleph_0$, then $2^{<\kappa} = \aleph_0$:
For any natural numbers k and n,

$$\text{nonspecial tree } \rightarrow (\omega + n)^2_k \, .$$

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Set $\kappa = \aleph_0$, then $2^{<\kappa} = \aleph_0$:
For any natural numbers k and n,

$$\text{nonspecial tree} \ \to (\omega + n)^2_k \,.$$

However, this case already follows from a stronger result:

Theorem (Todorčević, 1985)

For all $\alpha < \omega_1$ and $k < \omega$ we have

$$\textit{nonspecial tree} \ \to (\alpha)^2_k \,.$$

(This itself is a generalization to trees of an earlier result of Baumgartner and Hajnal, 1973, for cardinals.)

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(This itself is a generalization to trees of an earlier result of Baumgartner and Hajnal, 1973, for cardinals.)
So the first case where we get something new is:
Let $\kappa = \aleph_1$, then $2^{<\kappa} = \mathfrak{c}$, but ξ must still be finite, so we have:
For any natural numbers k and n,

$$\text{non-}\mathfrak{c}\text{-special tree } \to (\omega_1 + n)^2_k .$$

What about the case when κ is a singular cardinal?

What about the case when κ is a singular cardinal?
Depending on the values of the continuum function, there may be some singular cardinals κ for which the sequence $\{2^\mu : \mu < \kappa\}$ stabilizes, in which case such κ would satisfy $\mathrm{cf}(2^{<\kappa}) \geq \kappa$, so the Main Theorem applies.
In this case, it is significant that the κ in the conclusion does not need to be weakened to $\mathrm{cf}(\kappa)$.
Of course, this cannot happen under GCH.

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Of course, this cannot happen under GCH.
We will now begin to prove the Main Theorem.

Fix ν and κ such that $\nu^{<\kappa} = \nu$, and a non-ν-special tree T. We will use elementary submodels to create certain algebraic structures on the given tree T.

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We will consider a node $t \in T$ and elementary submodel $N \prec H(\theta)$ (for large enough θ) such that:

1. $T \in N$;
2. $t\downarrow \subseteq N$;
3. $t \notin N$;
4. Eligibility Condition: $\nexists B \in N[t\downarrow \subseteq B$ and $t \notin B]$;
5. $|N| = \nu$;
6. $[N]^{<\kappa} \subseteq N$.

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How do we know such nodes and models exist?
We'll see later.
For now, suppose we can fix such t and N.

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is the corresponding maximal (proper) ideal in the same set algebra.
What we really want are algebraic structures on $t{\downarrow}$ determined by N.

So we define a collapsing function

$$\pi : \mathcal{P}(T) \cap N \to \mathcal{P}(t\downarrow)$$

by setting

$$\pi(B) = B \cap t\downarrow.$$

Define $\mathcal{A} = \text{range}(\pi)$. So $\mathcal{A}$ is a set algebra over $t\downarrow$.

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Then set

$$\mathcal{G} = \{\pi(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \notin B\}.$$

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Then set

$$\mathcal{G} = \{\pi(B) : B \in \mathcal{P}(T) \cap N \text{ and } t \notin B\}.$$

Claim

The set $\mathcal{G}$ is a maximal proper ideal in the set algebra $\mathcal{A}$.

Proof.

If $\mathcal{G}$ were not proper (meaning $t\downarrow \in \mathcal{G}$), there would be some $B \in N$ with $t \notin B$ such that $\pi(B) = t\downarrow$. But then $t\downarrow \subseteq B$, contradicting the eligibility condition. □

We now consider the ideal on $t{\downarrow}$ generated by $\mathcal{G}$:
Define

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq Y \text{ for some } Y \in \mathcal{G}\}\,.$$

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Claim

Facts about $I_{N,t}$:

- *$I_{N,t}$ is a proper ideal on $t{\downarrow}$.*

$$I_{N,t} = \{X \subseteq t{\downarrow} : X \subseteq A \textit{ for some } A \in N \textit{ with } t \notin A\}\,.$$
$$I^{+}_{N,t} = \{X \subseteq t{\downarrow} : \forall B \in N\, [X \subseteq B \Rightarrow t \in B]\}\,.$$
$$I^{*}_{N,t} = \{X \subseteq t{\downarrow} : X \supseteq B \cap t{\downarrow} \textit{ for some } B \in N \textit{ with } t \in B\}\,.$$

- *If $X \subseteq s{\downarrow}$ for some $s <_T t$, then $X \in I_{N,t}$.*

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- *If $X \subseteq s{\downarrow}$ for some $s <_T t$, then $X \in I_{N,t}$.*

Furthermore, since we assumed $[N]^{<\kappa} \subseteq N$, all of our structures defined using N will be κ-complete, including $I_{N,t}$.

Definition

If $c : [T]^2 \to \mu$ is a colouring, where μ is some cardinal, and $\chi < \mu$ is some ordinal (colour), and $t \in T$, define

$$c_\chi(t) = \{s <_T t : c\{s, t\} = \chi\}.$$

Definition

If $c : [T]^2 \to \mu$ is a colouring, where μ is some cardinal, and $\chi < \mu$ is some ordinal (colour), and $t \in T$, define

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Lemma

Suppose we have cardinals μ and κ, colouring $c : [T]^2 \to \mu$, some colour $\chi < \mu$, and some node $t \in T$.
Suppose also that $N \prec H(\theta)$ is an elementary submodel such that $T, c, \chi \in N$, and also $[N]^{<\kappa} \subseteq N$, and $t\downarrow \subseteq N$.
If $X \subseteq c_\chi(t)$ is such that $X \in I^+_{N,t}$, then there is $Y \subseteq X$ such that Y is χ-homogeneous and $|Y| = \kappa$.

PROOF:
We will recursively construct a χ-homogeneous set

$$Y = \langle y_\eta \rangle_{\eta < \kappa} \subseteq X,$$

of order type κ, as follows:
Fix some ordinal $\eta < \kappa$, and suppose we have constructed χ-homogeneous

$$Y_\eta = \langle y_\iota \rangle_{\iota < \eta} \subseteq X$$

of order type η. We need to choose $y_\eta \in X$ such that $Y_\eta < \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous.
Since $Y_\eta \subseteq X \subseteq t{\downarrow} \subseteq N$ and $|Y_\eta| < \kappa$, the hypothesis that $[N]^{<\kappa} \subseteq N$ gives us $Y_\eta \in N$. Define

$$Z = \{s \in T : (\forall y_\iota \in Y_\eta)\,[y_\iota <_T s \text{ and } c\,\{y_\iota, s\} = \chi]\}\,.$$

Since Z is defined from parameters T, Y_η, c, and χ that are all in N, it follows by elementarity of N that $Z \in N$, and $Z \cap t\!\downarrow \in \mathcal{A}$. Since $Y_\eta \subseteq X \subseteq c_\chi(t)$, it follows from the definition of Z that $t \in Z$. But then we have $Z \cap t\!\downarrow \in \mathcal{G}^* \subseteq I^*_{N,t}$. By assumption we have $X \in I^+_{N,t}$. The intersection of a filter set and a co-ideal set must be in the co-ideal, so we have $X \cap Z \in I^+_{N,t}$. In particular, this set is not empty, so we choose $y_\eta \in X \cap Z$. Because $y_\eta \in Z$, we have $Y_\eta <_T \{y_\eta\}$ and $Y_\eta \cup \{y_\eta\}$ is χ-homogeneous, as required. □

We now generalize Kunen's definition of a nice chain of elementary submodels of $H(\theta)$:

Definition

Let λ be any regular uncountable cardinal, and let T be a tree of height λ. The collection $\langle W_t \rangle_{t \in T}$ is called a nice collection of sets indexed by T if:

1. For each $t \in T$, $|W_t| < \lambda$;
2. For $s, t \in T$ with $s <_T t$, $W_s \subseteq W_t$;
3. The collection is continuous (with respect to its indexing), meaning that for all $t \in T$ with height a limit ordinal,

$$W_t = \bigcup_{s <_T t} W_s.$$

Definition (continued)

Suppose furthermore that $\theta \geq \lambda$ is a regular cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a nice collection of elementary submodels of $H(\theta)$ indexed by T if, in addition to being a nice collection of sets as above, we have:

4. For each $t \in T$, $N_t \prec H(\theta)$;
5. For each $t \in T$, $t{\downarrow} \subseteq N_t$;
6. For $s, t \in T$ with $s <_T t$, $N_s \in N_t$.

Definition (continued)

Suppose furthermore that $\theta \geq \lambda$ is a regular cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a nice collection of elementary submodels of $H(\theta)$ indexed by T if, in addition to being a nice collection of sets as above, we have:

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If κ is an infinite cardinal, then we say $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels if, in addition to the above conditions, we have

7. For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.

Definition (continued)

Suppose furthermore that $\theta \geq \lambda$ is a regular cardinal. The collection $\langle N_t \rangle_{t \in T}$ is called a nice collection of elementary submodels of $H(\theta)$ indexed by T if, in addition to being a nice collection of sets as above, we have:

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7. For $s, t \in T$ with $s <_T t$, $[N_s]^{<\kappa} \subseteq N_t$.

If $\langle M_t \rangle_{t \in T}$ and $\langle N_t \rangle_{t \in T}$ are two nice collections of sets, then we say that $\langle N_t \rangle_{t \in T}$ is a fattening of $\langle M_t \rangle_{t \in T}$ if for all $t \in T$ we have $M_t \subseteq N_t$.

Lemma

Suppose λ is any regular uncountable cardinal, T is a tree of height λ, and $\theta \geq \lambda$ is a regular cardinal such that $T \subseteq H(\theta)$. Fix $X \subseteq H(\theta)$ with $|X| < \lambda$. Then:

1. *There is a nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$ such that $X \subseteq N_\emptyset$ (and therefore $X \subseteq N_t$ for every $t \in T$).*
2. *Given any nice collection $\langle M_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$, we can fatten the collection to include X, that is, we can construct another nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels of $H(\theta)$, that is a fattening of $\langle M_t \rangle_{t \in T}$, such that $X \subseteq N_\emptyset$.*
3. *If κ is an infinite cardinal such that for all $\nu < \lambda$ we have $\nu^{<\kappa} < \lambda$, then the nice collections we construct in parts (1) and (2) can be κ-very nice collections.*

Lemma

Suppose T is any tree, κ is any infinite cardinal, and $\theta > |T|$ is a regular cardinal. Suppose $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then for every $t \in T$, if $\mathrm{cf}(\mathrm{ht}(t)) \geq \kappa$ *then* $[N_t]^{<\kappa} \subseteq N_t$.

Lemma

Suppose T is any tree, κ is any infinite cardinal, and $\theta > |T|$ is a regular cardinal. Suppose $\langle N_t \rangle_{t \in T}$ is a κ-very nice collection of elementary submodels of $H(\theta)$. Then for every $t \in T$, if $\mathrm{cf}(\mathrm{ht}(t)) \geq \kappa$ then $[N_t]^{<\kappa} \subseteq N_t$.

Recall the earlier theorem that for a non-ν-special tree T, we have $T \upharpoonright S^{\nu^+}_{\mathrm{cf}(\nu)}$ is a stationary subtree.

Since we have $\mathrm{cf}(\nu) \geq \kappa$, this will give us a stationary subtree of nodes t whose associated models N_t satisfy $[N_t]^{<\kappa} \subseteq N_t$.

Proof.

Fix $t \in T$ such that $\mathrm{cf}(\mathrm{ht}(t)) \geq \kappa$. Fix a cardinal $\mu < \kappa$, and some collection

$$\mathcal{C} = \langle A_\iota \rangle_{\iota < \mu} \in [N_t]^\mu .$$

For each ordinal $\iota < \mu$, we have $A_\iota \in N_t$. Since $\mathrm{cf}(\mathrm{ht}(t)) \geq \kappa$, t must be a limit node, so since the collection of models is continuous, we have $A_\iota \in N_{s_\iota}$ for some $s_\iota <_T t$. Then define

$$s = \sup_{\iota < \mu} s_\iota \text{ (where the sup is taken along the chain } t{\downarrow}\text{).}$$

Since each $s_\iota <_T t$ and $\mu < \kappa \leq \mathrm{cf}(\mathrm{ht}(t))$, we have $s <_T t$. We then have, since the collection is κ-very nice,

$$\mathcal{C} \in [N_s]^\mu \subseteq [N_s]^{<\kappa} \subseteq N_t ,$$

as required. □

Recall the earlier eligibility condition for nodes and models. Given a nice collection of sets $\langle W_t \rangle_{t \in T}$, we will say that the node $t \in T$ is eligible if t and W_t satisfy the eligibility condition described earlier, that is,

$$\nexists B \in W_t [t{\downarrow} \subseteq B \text{ and } t \notin B].$$

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Lemma

Suppose ν is any infinite cardinal, and let T be a tree of height ν^+. Suppose $\langle W_t \rangle_{t \in T}$ is a nice collection of sets. Then the set of ineligible nodes is a nonstationary subtree of T.

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Suppose ν is any infinite cardinal, and let T be a tree of height ν^+. Suppose $\langle W_t \rangle_{t \in T}$ is a nice collection of sets. Then the set of ineligible nodes is a nonstationary subtree of T.

PROOF:

For any fixed set B, the set $\{t \in T : t\downarrow \subseteq B \text{ and } t \notin B\}$ is an antichain. For any $s \in T$, we have $|W_s| \leq \nu$, so it follows that

$$\bigcup_{B \in W_s} \{t \in T : t\downarrow \subseteq B \text{ and } t \notin B\}$$

is a union of $\leq \nu$ antichains, that is, it is a ν-special subtree.

Since the set of successor nodes is always a nonstationary subtree, we can consider only limit nodes. Suppose t is a limit node. Then by continuity of the nice collection $\langle W_t \rangle_{t \in T}$, if $B \in W_t$ then $B \in W_s$ for some $s <_T t$. So

$$
\begin{aligned}
&\{\text{limit ineligible nodes } t\} \\
&= \{\text{limit } t \in T : \exists s <_T t \exists B \in W_s \, [t{\downarrow} \subseteq B \text{ and } t \notin B]\} \\
&= \bigtriangledown_{s \in T} \{\text{limit } t \in T : \exists B \in W_s \, [t{\downarrow} \subseteq B \text{ and } t \notin B]\} \\
&= \bigtriangledown_{s \in T} \bigcup_{B \in W_s} \{\text{limit } t \in T : [t{\downarrow} \subseteq B \text{ and } t \notin B]\} \in NS_\nu^T,
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and it follows that the set of ineligible nodes is in NS_ν^T, as required. □

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and it follows that the set of ineligible nodes is in NS_ν^T, as required. □

From the previous two lemmas, we get a stationary subtree $S \subseteq T$ such that every $t \in S$ is eligible and satisfies $[N_t]^{<\kappa} \subseteq N_t$.

Given our tree T and a colouring $c : [T]^2 \to k$, we now fix a κ-very nice collection $\langle N_t \rangle_{t \in T}$ of elementary submodels with $T, c \in N_\emptyset$.

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Recall that we defined an ideal $I_{N,t}$ on $t\downarrow$ determined by node $t \in T$ and model $N \prec H(\theta)$.
Now that we have fixed a nice collection $\langle N_t \rangle_{t \in T}$, we will write I_t instead of $I_{N_t,t}$, because the node t determines the model N_t and therefore the ideal.

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Definition

Let $S \subseteq T$ and choose $t \in T$. If $S \cap t\downarrow \in I_t^+$ then t is called a reflection point of S.

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Definition

Let $S \subseteq T$ and choose $t \in T$. If $S \cap t\downarrow \in I_t^+$ then t is called a reflection point of S.

Some easy facts:

Fact

- *If $t \in T$ is a reflection point of some $S \subseteq T$ then t is eligible.*
- *If $t \in T$ is a reflection point of S then t is a limit point of S.*
- *If $R \subseteq S \subseteq T$ and $t \in T$ is a reflection point of R then s is a reflection point of S.*

We want to be able to know when some eligible $t \in T$ is a reflection point of some subtree $S \subseteq T$. Is it enough to assume that $t \in S$? If $S \in N_t$ and $t \in S$ is eligible, then we have $S \cap t{\downarrow} \in I_t^+$, so that t is a reflection point of S. Furthermore, if $S \in N_t$ for some $t \in T$, we know precisely which $u \in t{\uparrow}$ are reflection points of S, namely those eligible $u >_T t$ such that $u \in S$. But what if $S \notin N_t$? Then we can't guarantee that every $t \in S$ is a reflection point of S, but we can get close. The following lemma will be applied several times throughout the proof of the theorem:

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Lemma

For any $S \subseteq T$, we have

$$\{t \in S : S \cap t\downarrow \in I_t\} \in NS_\nu^T.$$

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Lemma

For any $S \subseteq T$, we have

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Proof Sketch.

The subtree that we claim to be nonstationary is included in the set of nodes that are ineligible after we fatten the models to contain S.

Definition

We define subtrees $S_n \subseteq T$, for $n \leq \omega$, by recursion on n:
First, define

$$S_0 = \left\{ t \in T : t \text{ is eligible and } [N_t]^{<\kappa} \subseteq N_t \right\}.$$

Then, for every $n < \omega$, define

$$S_{n+1} = \left\{ t \in S_n : S_n \cap t{\downarrow} \in I_t^+ \right\}.$$

Finally, define

$$S_\omega = \bigcap_{n<\omega} S_n.$$

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Finally, define

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Lemma

1. *For all $n < m \leq \omega$, each $t \in S_m$ is a reflection point of S_n, and therefore also a limit point of S_n.*
2. *For all $n < m \leq \omega$, the set $S_n \setminus S_m$ is nonstationary in T.*
3. *For all $n \leq \omega$, S_n is a stationary subtree of T.*

We will now define ideals $I(t, \sigma)$ and $J(t, \sigma)$ on $t{\downarrow}$, for certain nodes $t \in T$ and finite sequences of colours $\sigma \in k^{<\omega}$. We follow the convention that properness is *not* required for a collection of sets to be called an ideal (or a filter). In fact, some of our ideals will not be proper.

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Though we define the ideals $I(t, \sigma)$ and $J(t, \sigma)$, our intention will be to focus on the corresponding co-ideals. As we shall see, for a set to be in some co-ideal $I^+(t, \sigma)$ implies that it will include homogeneous sets of size κ for every colour in the sequence σ. This gives us the flexibility to choose later which colour in σ will be used when we combine portions of such sets to get a set of order type $\kappa + \xi$, homogeneous for the colouring c.

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We will define ideals $J(t, \sigma)$ and $I(t, \sigma)$ jointly by recursion on the length of the sequence σ. The collection $J(t, \sigma)$ will be defined for all $\sigma \in k^{<\omega}$ but only when $t \in S_{|\sigma|}$, while the collection $I(t, \sigma)$ will be defined only for nonempty sequences σ but for all $t \in S_{|\sigma|-1}$. (When $\sigma \in k^n$ we say $|\sigma| = n$.)

Definition

- Begin with the empty sequence, $\sigma = \langle\rangle$. For $t \in S_0$, we define

$$J(t, \langle\rangle) = I_t.$$

- Fix $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|}$, and assume we have defined $J(t,\sigma)$. Then, for each colour $i < k$, we define $I(t, \sigma^\frown \langle i\rangle) \subseteq \mathcal{P}(t{\downarrow})$ by setting, for $X \subseteq t{\downarrow}$,

$$X \in I(t, \sigma^\frown \langle i\rangle) \iff X \cap c_i(t) \in J(t,\sigma).$$

- Fix $\sigma \in k^{<\omega}$ with $\sigma \neq \emptyset$, and assume we have defined $I(s,\sigma)$ for all $s \in S_{|\sigma|-1}$. Fix $t \in S_{|\sigma|}$. We define $J(t,\sigma) \subseteq \mathcal{P}(t{\downarrow})$ by setting, for $X \subseteq t{\downarrow}$,

$$X \in J(t,\sigma) \iff \{s \in S_{|\sigma|-1} \cap t{\downarrow} : X \cap s \in I^+(s,\sigma)\} \in I_t.$$

Fact

For each sequence σ and each relevant t, the collections $I(t, \sigma)$ and $J(t, \sigma)$ are κ-complete ideals on $t{\downarrow}$ (though not necessarily proper).

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Lemma

For all nonempty $\sigma \in k^{<\omega}$ and all $t \in S_{|\sigma|-1}$, we have

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and equivalently,

$$I_t^+ \supseteq I^+(t, \sigma), \text{ and } I_t^* \subseteq I^*(t, \sigma).$$

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Lemma

Fix nonempty $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|-1}$. If $X \subseteq t{\downarrow}$ and $X \in I^+(t,\sigma)$, then for all $j \in \mathrm{range}(\sigma)$ there is $W \subseteq X$ such that $|W| = \kappa$ and W is j-homogeneous.

Definition

For any ordinal ρ and sequence $\sigma \in k^{<\omega}$, we consider chains in T of order type $\rho^{|\sigma|}$, and we define, by recursion over the length of σ, what it means for such a chain to be (ρ, σ)-good:

- Beginning with the empty sequence $\langle\rangle$, we say that every singleton set is $(\rho, \langle\rangle)$-good.
- Fix a sequence $\sigma \in k^{<\omega}$, and suppose we have already decided which chains in T are (ρ, σ)-good. Fix a colour $i < k$. We say that a chain $X \subseteq T$ of order type $\rho^{|\sigma|+1}$ is $(\rho, \sigma^\frown \langle i \rangle)$-good if

$$X = \bigcup_{\eta < \rho} X_\eta$$

where the sequence $\langle X_\eta : \eta < \rho \rangle$ satisfies the following conditions:

1. for each $\eta < \rho$, the chain X_η is (ρ, σ)-good,
2. for each $\iota < \eta < \rho$, we have $X_\iota < X_\eta$, and
3. for each $\iota < \eta < \rho$,

$$c''(X_\iota \otimes X_\eta) = \{i\}.$$

Lemma

Fix $\sigma \in k^{<\omega}$ *and ordinal* ρ. *If* X *is* (ρ, σ)*-good, then for all* $j \in \text{range}(\sigma)$ *there is* $Y \subseteq X$ *such that* Y *is* j*-homogeneous for* c *and has order-type* ρ.

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Lemma

Fix nonempty $\sigma \in k^{<\omega}$ *and* $t \in S_{|\sigma|-1}$*. If* $X \in I^+(t, \sigma)$ *then for all* $\rho < \kappa$ *there is* $Y \subseteq X$ *that is* (ρ, σ)*-good.*

Lemma

Fix $\sigma \in k^{<\omega}$ and ordinal ρ. If X is (ρ, σ)-good, then for all $j \in \text{range}(\sigma)$ there is $Y \subseteq X$ such that Y is j-homogeneous for c and has order-type ρ.

Lemma

Fix nonempty $\sigma \in k^{<\omega}$ and $t \in S_{|\sigma|-1}$. If $X \in I^+(t, \sigma)$ then for all $\rho < \kappa$ there is $Y \subseteq X$ that is (ρ, σ)-good.

Lemma

Fix $\sigma \in k^{<\omega}$ and $m < \omega$. If ρ and ξ are any two ordinals such that

$$\rho \to (\xi)^1_m,$$

if $X \subseteq T$ is (ρ, σ)-good, and $g : X \to m$ is some colouring, then there is some $Y \subseteq X$, homogeneous for g, such that Y is (ξ, σ)-good.

Lemma

Fix $m < \omega$. For any infinite cardinal τ, and any ordinal $\xi < \tau$, there is some ordinal ρ with $\xi \leq \rho < \tau$ such that

$$\rho \to (\xi)^1_m .$$

Proof.

To see this, consider two cases:

- *Suppose $\tau = \omega$. In this case, $\xi < \tau$ is necessarily finite, so we can let $\rho = (\xi - 1) \cdot m + 1$.*
- *Otherwise, τ is an uncountable cardinal. For $\xi < \tau$, let $\rho = \omega^\xi$ (ordinal exponentiation). We clearly have $\xi \leq \rho < \tau$. Any ordinal power of ω is indecomposable, that is,*

$$(\forall m < \omega) \left[\omega^\xi \to \left(\omega^\xi \right)^1_m \right] ,$$

giving us a homogeneous chain even longer than required. □

From here onward, we will generally be working within the subtree

$$S_\omega = \bigcap_{n<\omega} S_n,$$

as defined earlier. Notice that if $t \in S_\omega$, then $I(t, \sigma)$ is defined for all nonempty $\sigma \in k^{<\omega}$.

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Definition

We begin by defining

$$\Sigma_0 = \left\{\sigma \in k^{<\omega} : \sigma \neq \emptyset \text{ and } \sigma \text{ is one-to-one}\right\}.$$

For a stationary subtree $S \subseteq S_\omega$ and $t \in S$, define

$$\Sigma(t,S) = \left\{\sigma \in \Sigma_0 : S \cap t{\downarrow} \in I^+(t,\sigma)\right\}.$$

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Each sequence $\sigma \in \Sigma_0$ has length $\leq k$, so the set Σ_0 is finite. Since for any t, S we have $\Sigma(t, S) \subseteq \Sigma_0$, there are only finitely many distinct sets $\Sigma(t, S)$.

Fact

For any stationary $R, S \subseteq S_\omega$, *if* $t \in R \subseteq S$ *then* $\Sigma(t, R) \subseteq \Sigma(t, S)$.

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For any stationary subtree $S \subseteq S_\omega$, recall that t is called a reflection point of S if $S \cap t{\downarrow} \in I_t^+$. Also recall that by a previous lemma, we have

$$\{t \in S : S \cap t{\downarrow} \in I_t\} \in NS_\nu^T.$$

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Fact

Fix any stationary subtree $S \subseteq S_\omega$. *If* t *is any reflection point of* S, *then we have*

$$\Sigma(t, S) \neq \emptyset.$$

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We would like to have a set on which Σ is constant:

Lemma

For every stationary set $R_0 \subseteq S_\omega$*, there are a stationary set* $R \subseteq R_0$ *and a fixed* $\Sigma \subseteq \Sigma_0$ *such that for all stationary* $S \subseteq R$ *we have*

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Any Σ obtained from this lemma must be nonempty. We explore the consequences of the sequence σ being maximal in Σ:

Lemma

For every stationary set $R_0 \subseteq S_\omega$*, there are a stationary set* $R \subseteq R_0$ *and a fixed* $\Sigma \subseteq \Sigma_0$ *such that for all stationary* $S \subseteq R$ *we have*

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Any Σ obtained from this lemma must be nonempty. We explore the consequences of the sequence σ being maximal in Σ:

Lemma

Suppose $S \subseteq S_\omega$ *is stationary, and there is* $\Sigma \subseteq \Sigma_0$ *such that*

$$\{t \in S : \Sigma(t, S) \neq \Sigma\} \in NS_\nu^T.$$

Suppose also that $\sigma \in \Sigma$ *is maximal by inclusion. Then there are* $u \in S$ *with* $\Sigma(u, S) = \Sigma$ *and stationary* $R \subseteq S$*, with* $\{u\} <_T R$*, such that*

$$(\forall t \in R)\left[S \cap u{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \in I(u, \sigma)\right].$$

Now it's time to put everything together to get the required homogeneous sets. Fix an ordinal $\xi < \log \kappa$, where $\log \kappa$ is the smallest cardinal τ such that $2^\tau \geq \kappa$. Recall that T is a non-ν-special tree, and $c : [T]^2 \to k$, and we need to find a set $X \subseteq T$ of order type $\kappa + \xi$ that is homogeneous for the partition c. The strategy will be to find some node $u \in T$ and chains $W, Y \subseteq T$ such that

$$W <_T \{u\} <_T Y,$$

where W has order type κ, Y has order type ξ, and $W \cup Y$ is homogeneous as required. That is, we require the chains W and Y to satisfy

$$[W]^2 \cup (W \otimes Y) \cup [Y]^2 \subseteq c^{-1}(\{i\})$$

for some $i < k$.

Recall that S_ω is stationary. We then fix stationary $S \subseteq S_\omega$ and $\Sigma \subseteq \Sigma_0$ such that for all stationary $R \subseteq S$ we have

$$\{t \in R : \Sigma(t, R) \neq \Sigma\} \in NS^T_\nu.$$

Then, $\Sigma \neq \emptyset$. Fix $\sigma \in \Sigma$ that is maximal by inclusion, and let $m = |\sigma|$.
We now apply the last lemma to S, Σ, and σ. This gives us $u \in S$ with $\Sigma(u, S) = \Sigma$ and stationary $R \subseteq S$, with $\{u\} <_T R$, such that

$$(\forall t \in R) \left[S \cap u{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(t) \in I(u, \sigma) \right].$$

Since $\Sigma(u, S) = \Sigma$, we have $\sigma \in \Sigma(u, S)$, meaning

$$S \cap u{\downarrow} \in I^+(u, \sigma).$$

Since $R \subseteq S$, by choice of S we have

$$\{t \in R : \Sigma(t, R) \neq \Sigma\} \in NS_\nu^T,$$

and R is stationary, so we can fix $t \in R$ such that $\Sigma(t, R) = \Sigma$, so that $\sigma \in \Sigma = \Sigma(t, R)$, giving

$$R \cap t{\downarrow} \in I^+(t, \sigma).$$

We have $\xi < \log \kappa \leq \kappa$, where of course $\log \kappa$ is infinite.
We obtain an ordinal ρ with $\xi \leq \rho < \log \kappa$ such that

$$\rho \to (\xi)^1_m .$$

We then obtain $Z \subseteq R \cap t{\downarrow}$ that is (ρ, σ)-good. Since $Z \subseteq R$, we have $\{u\} <_T Z$ and for every $s \in Z$ we have

$$S \cap u{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(s) \in I(u, \sigma).$$

Since Z is (ρ, σ)-good, it has order type ρ^m, and therefore $|Z| = |\rho^m| < \log \kappa \leq \kappa$. Since $I(u, \sigma)$ is a κ-complete ideal, it follows that

$$\bigcup_{s \in Z} \left(S \cap u{\downarrow} \cap \bigcup_{i \notin \mathrm{range}(\sigma)} c_i(s) \right) \in I(u, \sigma),$$

or

$$S \cap u{\downarrow} \cap \bigcup_{s \in Z} \left(\bigcup_{i \notin \text{range}(\sigma)} c_i(s) \right) \in I(u, \sigma).$$

We now let

$$H = S \cap u{\downarrow} \setminus \bigcup_{s \in Z} \left(\bigcup_{i \notin \text{range}(\sigma)} c_i(s) \right),$$

and since $S \cap u{\downarrow} \in I^+(u, \sigma)$, it follows that

$$H \in I^+(u, \sigma).$$

We can also write

$$H = \{ r \in S \cap u{\downarrow} : (\forall s \in Z) \, [c(\{r, s\}) \in \text{range}(\sigma)] \}.$$

For each $r \in H$, we define a function $g_r : Z \to \text{range}(\sigma)$ by setting, for each $s \in Z$,

$$g_r(s) = c(\{r, s\}).$$

How many different functions from Z to range(σ) can there be?
At most $|\sigma|^{|Z|}$. But $|Z| < \log \kappa$, so $|\sigma|^{|Z|} < \kappa$.
For each function $g : Z \to \text{range}(\sigma)$, define

$$H_g = \{r \in H : g_r = g\}.$$

There are fewer than κ such sets, and their union is all of H, which is in the κ-complete co-ideal $I^+(u, \sigma)$, so there must be some function g such that $H_g \in I^+(u, \sigma)$. Fix such a function $g : Z \to \text{range}(\sigma)$.
We then apply a previous Lemma to the colouring g, and we obtain $Z' \subseteq Z$, homogeneous for g, that is (ξ, σ)-good. That is, we have a (ξ, σ)-good $Z' \subseteq Z$ and a fixed colour $i \in \text{range}(\sigma)$ such that for all $s \in Z'$ we have $g(s) = i$. But this means that for all $r \in H_g$ and all $s \in Z'$ we have

$$c(\{r, s\}) = g_r(s) = g(s) = i,$$

showing that $H_g \otimes Z' \subseteq c^{-1}(\{i\})$.

Now Z' is (ξ, σ)-good and $i \in \text{range}(\sigma)$, so we fix $Y \subseteq Z'$ that is i-homogeneous for c and has order type ξ.
Also, we get $W \subseteq H_g$ such that $|W| = \kappa$ and W is i-homogeneous for c.
So then $W \cup Y$ is i-homogeneous of order type $\kappa + \xi$, as required.
This completes the proof of the theorem.