On the Borel Complexity of Characterizable Subgroups

joint work with D. Impieri

Sao Sebastiao, Brazil, August 16, 2013

Dedicated to Ofelia T. Alas on the occasion of her 70th birthday
Theorem (Kronecker (a special case))

For every irrational \( \alpha \in [0, 1] \) the set of all multiples \( \{n\alpha : n \in \mathbb{Z}\} \) is dense in \( \mathbb{R} \) modulo 1.

\( \mathcal{Z} \) – infinite strictly increasing sequences \( S = (u_n) \) of integers, 
\[ W_S = \{ \alpha \in [0, 1] : (u_n\alpha) \text{ is uniformly distributed mod 1} \} \]
for \( S \in \mathcal{Z} \) (where “uniformly distributed” means that
\[ \lim_{m} \frac{|\{ n \in \mathbb{N} : 1 \leq n \leq m \text{ and } a_n\alpha \in \Delta \}|}{m} = \mu(\Delta) \]
for every subinterval \( \Delta \subseteq [0, 1] \).

Theorem (Weyl 1916)

(a) If \( u_n = P(n) \) is a polynomial function of \( n \), then \( W_S \) contains all irrational \( \alpha \in [0, 1] \).
(b) \( W_S \) has measure 1 for every \( S \in \mathcal{Z} \).
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$W_S$ need not contain all irrational $\alpha \in [0,1]$ in (b):

**Example** (the sequence of factorials)

If $S = (n!)$, then

$$[0, 1] \ni \alpha = e - 2 = \sum_{n=2}^{\infty} \frac{1}{n!} \notin W_S$$

as $\frac{1}{n+1} < n!e < \frac{2}{n+1}$ (mod 1), so $n!e \to 0 \mod 1$.

**Example** (The Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$)

If $S = (f_n)$, then $\alpha = \frac{1 + \sqrt{5}}{2} \notin W_S$ as $f_n \alpha \to 0 \mod 1$ ($\alpha - 1 \in [0,1]$)

Indeed, $\alpha = \frac{1}{1+\alpha} = \frac{1}{1+\frac{1}{\frac{1}{1+\frac{1}{1+\ldots}}}} =: [0; 1, 1, \ldots]$ with convergents $\frac{f_{n-1}}{f_n}$,

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Replace reals mod 1 by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ written additively, with norm $\|x\| = \text{distance to the closest integer for } x \in \mathbb{R}$, Haar measure $\mu$.

**Definition (a set of singular points in Weyl’s theorem)**

Let $\Gamma_S = \{ x \in \mathbb{T} : u_n \alpha \to 0 \}$ for $S \in \mathbb{Z}$.

$\Gamma_S$ is related also to trigonometric series (Arbault sets).

**Lemma (Properties of the sets $\Gamma_S$)**

(a) $\Gamma_S$ is a (proper) subgroup of $\mathbb{T}$;
(b) $\Gamma_S = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n>k} \{ x \in \mathbb{T} : \|u_n \alpha\| \leq 1/m \}$ is a Borel set;
(c) $\mu(\Gamma_S) = 0$.

Proof. (a) – (b) The closed set $F_k = \bigcap_{n>k} \{ x \in \mathbb{T} : \|u_n \alpha\| \leq 1/4 \}$ has $\text{Int}(F_k) = \emptyset$ as $u_n \to \infty$, so by Baire category theorem $\Gamma_S \subseteq \bigcup_{k=1}^{\infty} F_k \neq \mathbb{T}$.
(c) $\mathbb{T} = n\mathbb{T}$ for all $n \in \mathbb{N}$, hence $[\mathbb{T} : \Gamma_S]$ is infinite and $\mu(\Gamma_S) = 0$.
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\theta = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} =: [0; a_1, a_2, \ldots],
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where \( a_n \in \mathbb{N} \) for \( n \geq 1 \). Let \( u_n, r_n \) be the denominators and the nominators of convergents of \( \theta \), then \( u_1 = 1, u_2 = a_2, r_1 = a_1, r_2 = a_1a_2 + 1 \) and

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\begin{align*}
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Then \(|\theta - \frac{r_n}{u_n}| < \frac{1}{u_n u_{n+1}}\) and \(|u_n \theta - r_n| < \frac{1}{u_n}\) for \( n \in \mathbb{N} \), so \( \theta \in \Gamma_{(u_n)} \).

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**Being a Borel set of $\mathbb{T}$, $\Gamma_S$ is either countable or $|\Gamma_S| = \aleph_1$.**

**Definition (Let $q_n = \frac{u_{n+1}}{u_n}$.)**

**Theorem (Egglestone 1952: $|\Gamma_S|$ depends on $q_n$)**

(a) $|\Gamma_S| = \aleph_1$ if $q_n \to \infty$;
(b) $\Gamma_S$ is countable if $(q_n)$ is bounded.

Neither (a) nor (b) are necessary conditions.

**Theorem (C. Kraaikamp and P. Liardet 1991)**

If $u_n = a_n u_{n-1} + u_{n-2}$ and $u_1 = 1$, then TFAE:
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If $H$ contains a compact subgroup $K$ of $G$ with countable torsion quotient $H/K$ (so that $H$ is even a countable union of compact subgroups [KK & DD]). Here “torsion” can be relaxed as the countable subgroup $H/K$ of the compact metrizable group $G/K$ is characterizable.

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We can assume wlog that $u_0 = 1$, then the homomorphism $d_S : H \to \mathbb{T}^\mathbb{N}$ defined by $d_S(x) = (u_n x) \in \mathbb{T}^\mathbb{N}$ is injective and $d_S(H) \subseteq \{(z_n) \in \mathbb{T}^\mathbb{N} : z_n \to 0\} =: c_0(\mathbb{T})$ and $H \to d_S(H)$ is a topological isomorphism when $H$ and $c_0(\mathbb{T})$ carry the induced topologies (from $\mathbb{T}$ and $\mathbb{T}^\mathbb{N}$, resp.).

The metric topology of $\mathbb{T}^\mathbb{N}$ determined by the sup-norm (i.e., $|z|_S = \sup_n \|z_n\|$ for $z = (z_n) \in \mathbb{T}^\mathbb{N}$) induces on $c_0(\mathbb{T})$ a Polish group topology finer than the product topology, so the topology $\tau_S$ of $H$ transferred to $H$ via $d_S$ is a finer Polish that does not depend of $S$ (i.e., if $H = \Gamma_{S'}$ as well, then $\tau_{S'} = \tau_S$).

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If $K$ is an uncountable Kronecker set of $\mathbb{T}$, then the $F_\sigma$-subgroup $\langle K \rangle$ is not Polishable (so, $\langle K \rangle$ is not characterizable).
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A non empty compact subset $K$ of an infinite compact metrizable abelian group $X$ is called a **Kronecker set**, if for every continuous function $f : K \to \mathbb{T}$ and $\varepsilon > 0$ there exists a $\nu \in \hat{X}$ such that

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Gabriyelyan extended Bíró’s theorem for infinite compact metrizable abelian group:

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Let $X$ be a compact metrizable abelian group. Then

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Example (uncountable characterizable $F_\sigma$-subgroups)

(a) [Gabriyelyan 2012] if $j : \mathbb{R} \hookrightarrow \mathbb{T}^2$ is a dense continuous monomorphism, then $j(\mathbb{R})$ is characterizable.
(b) if a subgroup $H$ of a compact metrizable group $G$ contains a compact subgroup $K$ such that $H/K$ is countable, then $H$ is characterizable.

Item (a) can be generalized as follows:

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Theorem (Gabriyelyan 2013)

All characterizable subgroups of compact metrizable abelian group are $F_\sigma$ iff $G$ has finite exponent.

Consequently, all compact metrizable abelian groups of infinite exponent contain a characterizable subgroup that is not an $F_\sigma$-set.

Example (characterizable, non-$F_\sigma$-subgroups of $\mathbb{T}$)

(a) [Bukovský, Kholshevikova, Repický 1994] $\Gamma_S$ is not an $F_\sigma$-subgroup of $\mathbb{T}$ for $S = (2^{2^n})$.

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Is there something common between (a) and (b)?

In both cases $u_n | u_{n+1}$ in $S = (u_n)$ and $q_n = \frac{u_{n+1}}{u_n} \to \infty$.

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Definition (Call a sequence $S = (u_n)$ of positive integers an arithmetic sequence (briefly, an a-sequence) if $u_n | u_{n+1}$ for all but finitely many $n$.)

Theorem (Impieri, DD 2013)

The following are equivalent for an a-sequence $S = (u_n) \in \mathbb{Z}$:

(a) $\Gamma_S \leq \mathbb{Q}/\mathbb{Z}$;
(b) $(q_n)$ is bounded;
(c) $\Gamma_S$ is countable;
(d) $\Gamma_S$ is an $F_\sigma$-set.
(e) $\tau_S$ is discrete.

(a) and (b) are specific properties of $\mathbb{T}$, while (c)–(e) can be discussed for every metrizable compact abelian group $G$ in place of $\mathbb{T}$ and (c) $\iff$ (e) holds true in general, (c) $\iff$ (d) is open even in $\mathbb{T}$. 

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(a) and (b) are specific properties of $\mathbb{T}$, while (c)—(e) can be discussed for every metrizable compact abelian group $G$ in place of $\mathbb{T}$ and (c) $\Leftrightarrow$ (e) holds true in general, (c) $\Leftrightarrow$ (d) is open even in $\mathbb{T}$. 

Joint work with D. Impieri

On the Borel Complexity of Characterizable Subgroups
Subgroups of $\mathbb{T}$ determined by a sequence

Definition (Call a sequence $S = (u_n)$ of positive integers an \textbf{arithmetic sequence} (briefly, an \textbf{a-sequence}) if $u_n | u_{n+1}$ for all but finitely many $n$.)

Theorem (Impieri, DD 2013)

The following are equivalent for an a-sequence $S = (u_n) \in \mathbb{Z}$:

(a) $\Gamma_S \leq \mathbb{Q}/\mathbb{Z}$;
(b) $(q_n)$ is bounded;
(c) $\Gamma_S$ is countable;
(d) $\Gamma_S$ is an $F_\sigma$-set.
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(a) and (b) are specific properties of $\mathbb{T}$, while (c)---(e) can be discussed for every metrizable compact abelian group $G$ in place of $\mathbb{T}$ and (c) $\iff$ (e) holds true in general, (c) $\iff$ (d) is open even in $\mathbb{T}$. 

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On the Borel Complexity of Characterizable Subgroups
Definition (Call a sequence \( S = (u_n) \) of positive integers an 
*arithmetic sequence* (briefly, an *a-sequence*) if \( u_n | u_{n+1} \) for all but 
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Theorem (Impieri, DD 2013)

*The following are equivalent for an a-sequence \( S = (u_n) \in \mathbb{Z}^*:*

(a) \( \Gamma_S \leq \mathbb{Q}/\mathbb{Z} \);

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(a) and (b) are specific properties of \( \mathbb{T} \), while (c) — (e) can be 
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\( \mathbb{T} \) and (c) \( \Leftrightarrow \) (e) holds true in general, (c) \( \Leftrightarrow \) (d) is open even in \( \mathbb{T} \).
Removing the hypothesis “a-sequence” in the theorem leads to

(a) $\Gamma_S \leq \mathbb{Q}/\mathbb{Z}$;
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Open questions

Question
If $\Gamma_S$ is an $F_\sigma$-set of $\mathbb{T}$ for some $S \in \mathbb{Z}^\mathbb{N}$, must $\Gamma_S$ be necessarily countable?

The answer is positive if $S$ is an a-sequence.

Question
If $H$ is a countable subgroup of $\mathbb{T}$, does there exist a characterizing sequence $S \in \mathbb{Z}^\mathbb{N}$ of $H$ with bounded sequence of ratios $(q_n)$?

Question
Does every Polishable $F_\sigma$-subgroup of $\mathbb{T}$ admit a characterizing sequence?

By Biro’s theorem, the answer is negative if we relax “Polishable”, by Gabriyelyan’s example, the answer is negative if we replace $\mathbb{T}$ be an arbitrary compact metrizable group.

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Weyl’s uniform distribution modulo 1 theorem
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