Simplicity and the quest for ultimate (mathematical) truth

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The mathematical story of infinity begins with:

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$$\emptyset$$

(The empty set)

Ordinals: the transfinite numbers

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If α is an ordinal then

$$\alpha + 1 = \alpha \cup \{\alpha\}$$

is the next largest ordinal.

V: The Universe of Sets

The power set

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Cumulative Hierarchy of Sets

The universe V of sets is generated by defining V_α by induction on the ordinal α:

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. *if α is a limit ordinal then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.*

- If X is a set then $X \in V_\alpha$ for some ordinal α.

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- The original principles (ZFC axioms) of Set Theory are naturally expanded by additional principles which assert the existence of "very large" infinite sets.
- These principles are called *large cardinal axioms*.

The hierarchy of large cardinal axioms–short version

- *There is a proper class of measurable cardinals*.
- *There is a proper class of strong cardinals*.
- *There is a proper class of Woodin cardinals*.
- *There is a proper class of superstrong cardinals*.

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- *There is a proper class of supercompact cardinals*.
- *There is a proper class of extendible cardinals*.
- *There is a proper class of huge cardinals*.
- *There is a proper class of* ω*-huge cardinals*.

The Continuum Hypothesis: CH

Theorem (Cantor)

The set $\mathbb{N}$ of all natural numbers and the set $\mathbb{R}$ of all real numbers do not have the same cardinality.

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The Continuum Hypothesis

Suppose $A \subseteq \mathbb{R}$ is infinite. Then either:

1. *A and $\mathbb{N}$ have the same cardinality, or*
2. *A and $\mathbb{R}$ have the same cardinality.*

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- The Continuum Hypothesis is a statement about just $V_{\omega+2}$.

Cohen's method

If M is a universe of Set Theory then M contains "blueprints" for virtual universes N which extend M. These blueprints can be constructed and analyzed from within M.

- *If M is countable then every blueprint constructed within M can be realized as genuine extension of M.*

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- *If M is countable then every blueprint constructed within M can be realized as genuine extension of M.*

- Cohen proved that **every** universe M contains a blueprint for an extension in which the Continuum Hypothesis is false.
- Cohen's method also shows that **every** universe M contains a blueprint for an extension in which the Continuum Hypothesis is true.

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Generic-Multiverse truth (Vision Statement)

A statement is a **Generic-Multiverse truth** *if it holds in each universe of the Generic-Multiverse.*

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Generic-Multiverse truth (Vision Statement)

A statement is a **Generic-Multiverse truth** *if it holds in each universe of the Generic-Multiverse.*

- *these are the* **universal laws** *of Set Theory.*

- These laws can be identified within each universe of the Generic-Multiverse.

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- **no matter how strong this notion is.**

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Claim

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So either:

- the Ω Conjecture is false; or
- the Generic-Multiverse approach is not an option.

Dealing with the ramifications of Cohen's method
Attempt 2: Refine the concept of set

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that X is logically definable in the structure $(X, \in)$ from parameters in X.

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- $\mathcal{P}_{\mathrm{Def}}(X)$ is the collection of all "simple" subsets of X
  - versus $\mathcal{P}(X)$ which is the collection of *all* subsets of X.

The effective cumulative hierarchy: L

Cumulative Hierarchy of Sets

The cumulative hierarchy is defined by induction on α as follows.

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. *if α is a limit ordinal then* $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

- *V is the class of all sets X such that $X \in V_\alpha$ for some α.*

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Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$.
2. $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$.
3. *if α is a limit ordinal then* $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

- *L is the class of all sets X such that $X \in L_\alpha$ for some α.*

The axiom: $V = L$

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- Suppose there is a Cohen-blueprint for $V = L$. Then:
  - the axiom $V = L$ must hold and the blueprint is trivial.

Theorem (after Gödel)

Assume $V = L$. Then V is the minimum universe of the Generic-Multiverse.

The axiom $V = L$ and large cardinals

Theorem (Scott)

Assume $V = L$. Then there are no measurable cardinals.

- **(There are no (genuine) large cardinals.)**

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(meta) Corollary

The axiom $V = L$ is false.

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(meta) Corollary

The axiom $V = L$ is false.

- The solution is obvious:

The Inner Model Program

Construct enlargements of L in which large cardinals can exist:

- *do this by using the large cardinals themselves to expand the definable powerset.*

A problem with this approach because of its incremental nature:

- *There can be no ultimate enlargement since its construction would require having identified* **all** *notions of higher infinity*
  - *so with every enlargement there is a generalization of Scott's Theorem.*
- *The program to understand V through enlargements of L can never succeed.*

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There is a very good candidate for the axiom $V = \text{Ultimate-}L$,

- **even though it is not yet known how to construct this enlargement.**

Consequences of the axiom $V = \mathrm{Ultimate}\text{-}L$

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

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Theorem ($V = \text{Ultimate-}L$)

V is the minimum universe of the Generic-Multiverse.

Technical afterword: the small print

The axiom for $V = \mathrm{Ultimate}\text{-}L$

- *There is a strong cardinal which is a limit of Woodin cardinals.*
- *For each Σ_3-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta} \models \varphi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

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$\mathrm{Ultimate}\text{-}L$ Conjecture

Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:

1. *N is a suitable extender model for δ is supercompact.*
2. *$N \subseteq \mathrm{HOD}$.*
3. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*