Effectivity in Abelian Group Theory

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Mal’cev 1962 A computable abelian group is **computably presented** if we have $G = (G, +, 0)$ has $+$ and $=$ computable functions/relations on $G = \mathbb{N}$.

**When** can an abelian group be computably presented? (Relative to an oracle) Is there any reasonable answer?

Do different computable presentations have different computable properties?

Mal’cev produced examples presentations of $\mathbb{Q}^\infty$ that were not computably isomorphic, as we see later.

Along with Rabin and Frölich and Shepherdson, began the theory of presentations of computable structures, though arguably back to Emmy Noether as recycled in van der Waerden (first edition).

See Metakies and Nerode “Effective Content of Field Theory”.
Describe computably presentable Abelian groups. Reduced $p$-groups classically described in terms of Ulm Invariants, can this be effectivized?

**Theorem (Khisamiev 1970’s, Ash-Knight-Oates 1980’s)**

A certain characterization of computable reduced abelian $p$-groups of finite Ulm type in terms of limitwise monotonic approximations of functions.

- Recall that a set $S$ is limitwise monotonic iff $S = \operatorname{ra}(f)$ for some computable $f = f(\cdot,\cdot)$, where for $\lim_s f(n,s)$ exists for all $n$, and $f(n,s+1) \geq f(n,s)$ for all $s$.

- Sometimes the function $f$ has only elements of $\omega$ in its range and sometimes for convenience we have $\infty$ there.

- Fact: the finite members of the range of one of these functions is a $\Sigma^0_2$ set.
Ulm’s Theorem

- $G$ is a $p$-group if each element has order $p^n$ for some $n$. $G$ is reduced if no element of infinite height. The height of $g$ is the largest $n$ with $p^n x = g$ having a solution (or $\infty$).

- **Ulm Sequence** $G_0 = G$, $G_{\alpha + 1} = pG_\alpha$, and for limit $G_\alpha = \bigcap_{\beta < \alpha} G_\beta$. There is some $\alpha = \lambda(G)$ with $G_\alpha = G_{\alpha + 1}$.

- This $\alpha = \lambda(G)$ is called the **length**. If $G$ is computable then $\alpha < \omega_1^{CK}$ by general results.

- Let $P = \{a \in G | pa = 0\}$ and considering $\frac{G_\beta \cap P}{G_{\beta + 1} \cap P}$ as a vector space over $\mathbb{Z}_p$, we get a sequence $(u_\beta(G))_{\beta < \alpha}$, called the **Ulm sequence**.

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**Theorem (Ulm, 1933)**

If $(u_\beta(G))_{\beta < \alpha}$ is a countable sequence of elements of $\omega \cup \{\infty\}$, then there is a countable group with this sequence iff (i) if $\alpha = \beta + 1$, $u(\beta) \neq 0$ and (ii) for any limit $\beta < \alpha$, there is an increasing $\beta_n \neq 0$ and $\beta_n \to \beta$. 
Theorem (Khisamiev; Ash, Knight, Oates)

Let $G$ be a countable reduced abelian $p$-group with length $\lambda(G) < \omega^2$, the $G$ has a computable copy iff

1. the relation $R_i = \{ (n, k) \mid u_{\omega.i+n}(G) \geq k \}$ is $\Sigma^0_{2i+2}$, and

2. There is a $\Delta^0_{2i+1}$ function such that for each $i$, $f_i(n, s)$ is a limitwise monotomic with finite limit $m$ and $u_{\omega.i+m}(G) \neq \emptyset$.

We remark that if we are given any length $\nu < \omega_1^{CK}$ and the $\Delta^0_{2i+1}$ functions uniformly, then we have a group $G$ corresponding to the functions.
Question (Khisameiv, Ash et al.) Does this hold for ordinals $\geq \omega^2$? If not what is a possible characterization?

The problem is that the proof is nonuniform, and works by induction on ordinals below $\omega^2$. It appears to lack uniformity.

Theorem (Downey, Menikov, Ng)

There is a computable abelian $p$-group of Ulm length $\omega^2$ which does not satisfy the uniform version of Khisamiev-Ash-Knight-Oates theorem. Therefore, their proof can not be pushed up to $\omega^2$.

Strangely, the proof filters through computable trees.

Laurel Rogers gave an analysis of Ulm’s Theorem in TAMS in the 1960’s demonstrating that you can obtain it via trees.

Question: Is there a computable reduced $p$-group with no corresponding computable tree? Conj Yes (Downey), No (Melnikov), No Clue (current state of affairs).
Rogers’ analysis

- $T = (\omega^{<\omega}, p, \emptyset)$, $p$ predecessor.
- $G(T)$ via $\emptyset = 0$, $pa = b$ iff $p(a) = b$ $b \in G(T)$ represented by $\sum_{i=1}^{n} k_ia_i$ with $a_i \in T$ and $k_i \in \omega$.
- (Rogers) If $T$ has no infinite branches then $G(T)$ is a reduced abelian $p$-group. The converse is also true.
- Trees are not unique, but there is an equivalence relation which is $T_1 \equiv T_1$, then $G(T_1) \cong G(T_2)$, and conversely. Equivalence relation $=$ sequences of “strippings”
- If $T$ is computable, so is $G(T)$. Open : Converse?
- The Ash, Knight, Oates proof shows how to construct a computable tree from the given information.
Problem [Khisamiev 1990’s] Describe computable groups of the form \( \bigoplus_{p \in P} Q^{(p)} \), where \( P \) is a set of primes, and
\[
Q^{(p)} = \left\{ \frac{n}{p^k} : n \in \mathbb{Z} \text{ and } k \in \mathbb{N} \right\}.
\]

Theorem (Khisamiev 2002)
The group \( G_P \) is computable with some extra condition if and only if \( P \) is not in a certain proper subclass of hh-immune sets.

Theorem (Downey, Goncharov, Knight et al. 2010)
The group \( G_P \) is computable if and only if \( P \) is \( \Sigma^0_3 \).
Computable Categoricity

- The effective classification tool.
- A computable structure $\mathcal{A}$ is computably categorical iff for all $\mathcal{B} \cong \mathcal{A}$, $\mathcal{A} \cong_{\text{computable}} \mathcal{B}$.
- relatively if it works for all oracles.
- There is a longstanding program to understand the relationship between $\cong$, $\cong_{\text{comp}}$, classical structure of $\mathcal{A}$ and logical structure of $\mathcal{A}$ in terms of definability.
- These all also have “higher up” versions, like $\Delta^0_\alpha$ categoricity, definability etc.

**Theorem (Goncharov, 1975)**

If $\mathcal{A}$ is 2-decidable, then $\mathcal{A}$ is computable cat iff it is relatively computably cat iff it has an effective naming, that is a c.e. Scott family of existential formulae with parameters $\bar{c}$, such that for all $\bar{a}, \bar{b}$ if they satisfy the same $\phi$, then they are automorphic.
More Recent Metatheorems

**Theorem (Downey, Kach, Lempp, Turetsky-Fund. Math)**

If $\mathcal{A}$ is 1-decidable and it is computably cat, then it is relatively $\Delta^0_2$ cat, as it has a $\Sigma_2$ Scott family.


For each $\alpha < \omega_1^{CK}$ there is a computably cat $\mathcal{A}$ which is not relatively $\Delta^0_\alpha$ cat.
Categoricity questions for abelian groups

- When we specialize to specific structures within which it is hard to code graphs questions become more complex. You actually have to do some algebra!
- This is not too hard if you have torsion, and in particular p-groups.
- These have proven useful in lots of areas, $\aleph_1$ categorical theories, equivalence relations, linear orderings, etc.

**Theorem (Goncharov, Smith)**

A computable p-group is computably categorical iff it can be written in one of the following forms.

1. $\left(\mathbb{Z}(p^\infty)\right)^\ell \oplus G$ for $\ell \in \omega \cup \{\infty\}$ and $G$ finite;
2. $\left(\mathbb{Z}(p^\infty)\right)^n \oplus \left(\mathbb{Z}_{p^k}\right)^\infty \oplus G$ where $G$ is finite, and $n, k \in \omega$. 
Torsion-Free Abelian Groups

- Here we will study torsion-free abelian groups. That is, they have no elements \( z \) with \( z^n \) trivial.
- Some kind of good behaviour.

**Theorem (Khisamiev)**

Every \( \Pi_{n+1}^0 \) presentable torsion-free abelian group is isomorphic to one which is \( \Delta_n^0 \)-presentable.

- In general the isomorphism problem is very complex:

**Theorem (Downey and Montalbán)**

The isomorphism problem for torsion-abelian groups is \( \Sigma_1^1 \) complete.
The idea of an invariant is that is ought to make the problem simpler.

Classical isomorphism is always $\Sigma^1_1$. “There is a function such that ....”

Invariants make this easier, you would expect. Dimension in a vector space makes the problem $\Delta^0_3$.

The point is that a $\Sigma^1_1$-completeness result result means that the cannot be reasonable invariants for the isomorphism problem.

This methodology understands invariant theory computationally.

There are other programmes like this as we now will see.
The Borel game

- This is related to work by the descriptive set theorists who seek to have a notion of Borel cardinality for isomorphism types.
- One class $\mathcal{C}$ is reducible to another $\mathcal{D}$ if there is a Borel mapping injectively taking the isomorphism types of $\mathcal{C}$ into $\mathcal{D}$.
- For example, rank 3 torsion free groups are above rank 2 groups here.
- H. Friedman, Kechris, Thomas, Hjorth etc.
The idea is to look at algebraically more tractable classes; this is what is done classically anyway.

Recall that if $G$ is a torsion-free then $G$ embeds into $\bigoplus_{i \in F}(\mathbb{Q}, +)$. The cardinality of the least such $F$ is called the (Prüfer) rank of $G$.

Khisamiev proved that there is an effective embedding.
The only groups we understand well are the rank one groups (and certain mild generalizations). If \( g \in G \), define \( t(g) = (a_1, a_1, \ldots) \) where \( a_i \in \{\infty\} \cup \omega \) and represents the maximum number of times \( p_i \) divides \( g \). Say that \( t(g) = t(h) \) if they are \( \equiv^* \), meaning that they must be \( \infty \) in the same places, but otherwise are finitely often different. Thus we can write \( t(G) \).

For example, a divisible group would have \((\infty, \infty, \ldots)\) as its type.

**Theorem (Baer, Levi)**

For rank 1 torsion-free abelian groups, \( G \cong H \) iff they have the same type.

One corollary is that if we consider \( T(G) = \{\langle x, y \rangle \mid x \leq t(G)_y \} \), then \( G \) is computably presentable iff \( T(G) \) is c.e.. (Mal’tsev)
Two Corollaries

- $G$ is a computably categorical torsion-free abelian group iff it has finite rank.

**Definition**

A structure $\mathcal{A}$ has a **degree** iff $\min\{\deg(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}$ exists.

- Strictly speaking, we would mean the isomorphism type here.
- (Jockusch) Can define **jump degree** by replacing $\deg(\mathcal{B})$ by $\deg(\mathcal{B})'$. The same for $\alpha$-th jump degree. **Proper** if no $\beta$-th jump degree for $\beta < \alpha$.
- (Coles, Downey and Slaman) Every torsion free abelian group of finite rank has first jump degree.
- (Anderson, Kach, Melnikov, Solomon) For each computable $\alpha$ and $a > 0^\alpha$ there is a torsion-free abelian group with proper $\alpha$-th jump degree $a$.
- Melnikov now has more ordinals.
The infinite rank case

- It could be hoped that if $G$ has infinite rank, then $G \cong \bigoplus_{i \in \omega} H_i$ with $H_i$ of rank one.
- Alas, this is not true, however, there is a class of groups for which this is true, called completely decomposable for which this does happen.
- What about categoricity for such groups?
- We cannot hope for computable categoricity, but can hope for things “higher up”.
The homogeneous case

- If $G \cong \oplus H$ for a fixed $H$ then $G$ is called homogeneous.

**Theorem (Downey and Melnikov)**

*Homogeneous computable torsion free abelian groups are $\Delta^0_3$ categorical.*

- The proof relies on a new notion of independence called $S$-independence generalizing a notion of Fuchs to sets $S$ of primes.
- $B$, a set of elements, is $S$-independent (in $G$) iff for all $p \in S$ and $b_1, \ldots, b_k \in G$,

\[ p \mid \sum_{i=1}^{k} m_i b_i \text{ implies } p \mid m_i \text{ for all } i. \]

- This bound is tight.
But when can it be $\Delta^0_2$ categorical?

- Recall that a set $S$ is called semilow if $\{e \mid W_e \cap S \neq \emptyset\} \leq \emptyset'$.  
- Semilow sets allow for a certain kind of local guessing, and aroze in (i) automorphisms of the lattice of computably enumerable sets (Soare) and in (ii) computational complexity as non-speedable ones. (Soare, Blum-Marques, etc.)

**Theorem (Downey and Melnikov)**

$G$ is $\Delta^0_2$ categorical iff the type of $H$ consists of only 0’s and $\infty$’s and the position of the 0’s is semilow.

- The proof is tricky and splits into 5 cases depending on “settling times”.
- We remark that this is one of the very few known examples of when $\Delta^0_2$ categoricity of structures has been classified.
The general completely decomposable case

**Theorem (Downey and Melnikov)**

A completely decomposable $G$ is $\Delta^0_5$ categorical. The bound is tight.

The proof uses methods from the homogeneous case, plus some new ideas. The sharpness is a coding argument. For sharpness we use copies of $\oplus_{i \in \omega} \mathbb{Z} \oplus \oplus_{i \in \omega} \mathbb{Q}^{(p)} \oplus \oplus_{i \in \omega} \mathbb{Q}^{(q)}$, where $p \neq q$ primes and $\mathbb{Q}^{(r)}$ denotes the additive group of the localization of $\mathbb{Z}$ by $r$. Then a relation $\theta$ on this group which is decidable in one copy and very bad in another. With some extra work we can also prove the following. We don’t know if the bound is sharp here.

**Corollary (Downey and Melnikov)**

The index set of completely decomposable groups is $\Sigma^0_7$. 
References

- Computable completely decomposable groups, (DM) to appear TAMS
- $\Delta^0_2$ Categorical Equivalence Relations and $p$-groups (DMN), in preparation
- Iterated effective embeddings of $p$-groups (DMN), in preparation
Thank You