Template iterations and maximal cofinitary groups

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Matsue, September 6th, 2013

- cofin(S_∞) is the set of cofinitary permutations in S_∞, i.e. permutations $\sigma \in S_\infty$ which have finitely many fixed points.
- A mapping $\rho : A \to S_\infty$ *induces* a cofinitary representation of $\mathbb{F}_A$ if the canonical extension of ρ to a homomorphism $\hat{\rho} : \mathbb{F}_A \to S_\infty$ is such that $\text{im}(\hat{\rho}) \subseteq \{I\} \cup \text{cofin}(S_\infty)$.

Forcing M.c.g.'s

Let A, X be disjoint non-empty sets and let $\rho : X \to S_\infty$ induce a cofinitary representation. Then $\mathbb{Q}_{A,\rho}$ is the poset of all (s, F) where $s \subseteq A \times \omega \times \omega$ is finite, s_a is a finite injection for all a and $F \subseteq \widehat{W}_{A \cup X}$ is finite. Define $(s, F) \leq_{\mathbb{P}_{A,\rho}} (t, E)$ iff

- $s \supseteq t$, $F \supseteq E$ and,
- for all $n \in \omega$ and $w \in E$, if $e_w[s, \rho](n) = n$ then already $e_w[t, \rho](n)\downarrow$ and $e_w[t, \rho](n) = n$.

If $X = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$. If A is clear from the context we just write $\mathbb{Q}$.

- $\mathbb{Q}_{A,\rho}$ is Knaster.
- Let G be $\mathbb{Q}_{A,\rho}$ generic and let $\rho_G : A \cup X \to S_\infty$ be a mapping extending ρ and such that for all $a \in A$

$$\rho_G(a) = \bigcup \{s_a : (\exists F \in \widehat{W}_{A \cup X})\ (s, F) \in G\}.$$

Then ρ_G induces a cofinitary representation of $A \cup X$ extending ρ.

Lemma: Complete Embeddings

Let $A_0 \cap A_1 = \emptyset$, $A = A_0 \cup A_1$ and let G be $\mathbb{Q}_{A,\rho}$-generic. Then

- $\mathbb{Q}_{A_0,\rho}$ is a complete suborder of $\mathbb{Q}_{A,\rho}$,
- $H = G \cap \mathbb{Q}_{A_0,\rho}$ is $\mathbb{Q}_{A_0,\rho}$-generic, $K = \{(s \restriction A_1, F) : (s, F) \in G\}$ is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$ and $\rho_G = (\rho_H)_K$.

Theorem

Let $|A| > \aleph_0$ and G be a $\mathbb{Q}_{A,\rho}$-generic over V. Then $\mathrm{im}(\rho_G)$ is a maximal cofinitary group in $V[G]$.

Proof

Let $z \notin X \cup A$, where $\rho : X \to S_\infty$. Suppose there in $V[G]$ there is $\sigma \in \mathrm{cofin}(S_\infty)$ such that $\rho'_G : A \cup X \cup \{z\} \to S_\infty$ defined by $\rho'_G \restriction X \cup A = \rho_G$, $\rho'_G(z) = \sigma$ induces a cofinitary representation. Let $\dot\sigma$ be a name for σ. Then there is $A_0 \subseteq A$ countable so that $\dot\sigma$ is a $\mathbb{Q}_{A_0,\rho}$-name and so $\sigma \in V[H]$, where $H = G \cap \mathbb{Q}_{A_0,\rho}$.

Let $a_1 \in A \setminus A_0$ and let K be defined as in the previous Lemma. Note that for every $N \in \omega$

$$D_{\sigma,N} = \{(s, F) \in \mathbb{Q}_{A_1,\rho_H} : (\exists n \geq N) s_{a_1}(n) = \sigma(n)\}$$

is dense in $\mathbb{Q}_{A_1,\rho_H}$ and so in $V[H][K]$

$$\exists^\infty n((\rho_H)_K(a_1)(n) = \sigma(n)).$$

However $(\rho_H)_K = \rho_G$, which contradicts that ρ'_G induces a cofinitary representation. $\square$

Lemma: Strong Embedding

Let $B, C \subseteq D$, $B \cap C = A$ be given set and $p \in \mathbb{Q}_{B,\rho}$. Then there is a condition $p_0 \in \mathbb{Q}_{A,\rho}$ such that whenever $q_0 \leq_{\mathbb{Q}_{C,\rho}} p_0$, then q_0 is compatible in $\mathbb{Q}_{D,\rho}$ with p.

- We say that $\mathbb{Q}_{B,\rho}$ has the strong embedding property and q_0 is called a strong reduction of p.
- If $C = A$, $B = D$ then the above gives in particular that $\mathbb{Q}_{A,\rho}$ is a complete suborder of $\mathbb{Q}_{B,\rho}$.

Definition: $\mathbb{L}$

$\mathbb{L}$ consists of pairs (σ, ϕ) such that $\sigma \in {}^{<\omega}({}^{<\omega}[\omega])$, $\phi \in {}^{\omega}({}^{<\omega}[\omega])$ such that $\sigma \subseteq \phi$, $\forall i < |\sigma|(|\sigma(i)| = i)$ and $\forall i \in \omega(|\phi(i)| \leq |\sigma|)$. The extension relation is defined as follows: $(\sigma, \phi) \leq (\tau, \psi)$ if and only if σ end-extends τ and $\forall i \in \omega\ (\psi(i) \subseteq \phi(i))$.

- A slalom is a function $\phi : \omega \to [\omega]^{<\omega}$ such that $\forall n \in \omega(|\phi(n)| \leq n)$. A slalom localizes a real $f \in {}^{\omega}\omega$ if there is $m \in \omega$ such that $\forall n \geq m(f(n) \in \phi(n))$.
- $\mathbb{L}$ adds a slalom which localizes all ground model reals.

- $\text{add}(\mathcal{N})$ is the least cardinality of a family $F \subseteq \omega^\omega$ such that no slalom localizes all members of F
- $\text{cof}(\mathcal{N})$ is the least cardinality of a family Φ of slaloms such that every real is localized by some $\phi \in \Phi$.
- $\mathfrak{a}_g \geq \text{non}(\mathcal{M})$.

In our intended forcing construction cofinally often we will force with the partial order $\mathbb{L}$, which using the above characterization will provide a lower bound for $\mathfrak{a}_g$.

Definition: σ-Suslin

Let $(\mathbb{S}, \leq_{\mathbb{S}})$ be a Suslin forcing notion, whose conditions can be written in the form (s, f) where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$. We will say that $\mathbb{S}$ is *n-Suslin* if whenever $(s, f) \leq_{\mathbb{S}} (t, g)$ and (t, h) is a condition in $\mathbb{S}$ such that

$$h \restriction n \cdot |s| = g \restriction n \cdot |s|$$

then (s, f) and (t, h) are compatible. A forcing notion is called *σ-Suslin*, if it is *n*-Suslin for some n.

- If $\mathbb{S}$ is n-Suslin and $m \geq n$, than $\mathbb{S}$ is also m-Suslin.
- Every σ-Suslin forcing notion is σ-linked and so has the Knaster property.
- Hechler forcing $\mathbb{H}$ is 1-Suslin, localization $\mathbb{L}$ is 2-Suslin.

Definition: Nice name for a real

Let $\mathbb{B}$ be a partial order and $y \in \mathbb{B}$. For each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain below y. We will say that the set $\{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ is *a nice name for a real below y* if

1. whenever $n \geq 1$, $b \in \mathcal{B}_n$ then $s(b) \in {}^n\omega$
2. whenever $m > n \geq 1$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and b, b' are compatible, then $s(b)$ is an initial segment of $s(b')$.

We can assume that all names for reals are nice and abusing notation we will write $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \in \omega}$.

Lemma: Canonical Projection of a name for a real

Let $\mathbb{A}$ be a complete suborder of $\mathbb{B}$, $y \in \mathbb{B}$ and x a reduction of y to $\mathbb{A}$. Let $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ be a nice name for a real below y. Then there is $\dot{g} = \{(a, s(a))\}_{a \in \mathcal{A}_n, n \geq 1}$, a $\mathbb{A}$-nice name for a real below x, such that for all $n \geq 1$, for all $a \in \mathcal{A}_n$, there is $b \in \mathcal{B}_n$ such that a is a reduction of b and $s(a) = s(b)$.

Whenever $\dot{f}, \dot{g}$ are as above, we will say that $\dot{g}$ is a canonical projection of $\dot{f}$ below x.

Definition: Good Suslin

Let $\mathbb{S}$ be a Suslin forcing notion, whose conditions can be written in the form (s, f) where $s \in {}^{<\omega}\omega$, $f \in {}^{\omega}\omega$. Then $\mathbb{S}$ is said to be *good* if whenever $\mathbb{A}$ is a complete suborder of $\mathbb{B}$, $x \in \mathbb{A}$ is a reduction of $y \in \mathbb{B}$ and $\dot{f}$ is a nice name for a real below y such that $y \Vdash_{\mathbb{B}} (\check{s}, \dot{f}) \in \dot{\mathbb{S}}$ for some $s \in {}^{<\omega}\omega$, there is a canonical projection $\dot{g}$ of $\dot{f}$ below x such that $x \Vdash (\check{s}, \dot{g}) \in \dot{\mathbb{S}}$.

$\mathbb{D}$ and $\mathbb{L}$ are good σ-Suslin forcing notions.

- Let $(L, \leq)$ be a linearly ordered set, $x \in L$. Then $L_x := \{y \in L : y < x\}$.
- If $L_0 \subseteq L$ and $A \subseteq L$, then the L_0-*closure* of A, $\mathrm{cl}_{L_0}(A)$, is the smallest set $B \supseteq A$ such that if $x \in B$ then $L_x \cap L_0 \subseteq B$.

Definition: Template

A *template* is a tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ where $L = L_0 \cup L_1$, $L_0 \cap L_1 = \emptyset$, $(L, \leq)$ is a linear order, $\mathcal{I} \subseteq \mathcal{P}(L)$, such that

- $\mathcal{I}$ is closed under finite intersections and unions, $\emptyset, L \in \mathcal{I}$.
- If $x, y \in L$, $y \in L_1$ and $x < y$ then $\exists A \in \mathcal{I}(A \subseteq L_y \wedge x \in A)$.
- If $A \in \mathcal{I}$, $x \in L_1 \backslash A$, then $A \cap L_x \in \mathcal{I}$.
- $\{A \cap L_1 : A \in \mathcal{I}\}$ is well-founded when ordered by inclusion.
- All $A \in \mathcal{I}$ are L_0-closed.

- Define $\mathrm{Dp} : \mathcal{I} \to \mathbb{ON}$ by letting $\mathrm{Dp}(A) = 0$ for $A \subseteq L_0$ and

$$\mathrm{Dp}(A) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathcal{I} \wedge B \cap L_1 \subset A \cap L_1\}.$$

  Let $\mathrm{Rk}(\mathcal{T}) = \mathrm{Dp}(L)$.
- For $A \subseteq L$ let

$$\mathcal{T}_A = ((A, \leq), \mathcal{I}{\restriction}A, L_0 \cap A, L_1 \cap A),$$

  where $\mathcal{I}{\restriction}A = \{A \cap B : B \in \mathcal{I}\}$. If $A \in \mathcal{I}$ then $\mathrm{Rk}(\mathcal{T}_A) = \mathrm{Dp}(A)$.
- For $x \in L$ let $\mathcal{I}_x = \{B \in \mathcal{I} : B \subseteq L_x\}$.

Definition: Iterating good σ-Suslin posets along a template and adding m.c.g.

Let $\mathbb{Q} = \mathbb{Q}_{L_0}$ the poset adding a m.c.g. with L_0-generators, $\mathbb{S}$ good σ-Suslin. $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is defined recursively:

If $\mathrm{Rk}(\mathcal{T}) = 0$, then $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S}) = \mathbb{Q}_{L_0}$. Let $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ be defined for all templates of rank $< \kappa$. Let $\mathrm{Rk}(\mathcal{T}) = \kappa$ and for all $B \in \mathcal{I}(\mathrm{Dp}(B) < \kappa)$ let $\mathbb{P}_B = \mathbb{P}(\mathcal{T}_B, \mathbb{Q}, \mathbb{S})$. Then

- $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ consists of all $P = (p, F^p)$ where p is a finite partial function with $\mathrm{dom}(p) \subseteq L$, $(p \restriction L_0, F^p) \in \mathbb{Q}$ and if $x_p \stackrel{\mathrm{def}}{=} \max\{\mathrm{dom}(p) \cap L_1\}$ is defined then $\exists B \in \mathcal{I}_{x_p}$ such that $P \restriction L_{x_p} = (p \restriction L_{x_p}, F^p) \in \mathbb{P}_B$, $p(x_p) = (\check{s}^p_x, \dot{f}^p_x)$, where $s^p_x \in {}^{<\omega}\omega$, $\dot{f}^p_x$ is a $\mathbb{P}_B$ name for a real and $(P \restriction L_{x_p}, p(x_p)) \in \mathbb{P}_B * \dot{\mathbb{S}}$.

Define $Q \leq_{\mathbb{P}} P$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, $(q\restriction L_0, F^q) \leq_{\mathbb{Q}} (p\restriction L_0, F^p)$, and if x_p is defined then either

- $x_p < x_q$ and $\exists B \in \mathcal{I}_{x_q}$ such that $P\restriction L_{x_q}, Q\restriction L_{x_q} \in \mathbb{P}_B$ and $Q\restriction L_{x_q} \leq_{\mathbb{P}_B} P\restriction L_{x_q}$, or
- $x_p = x_q$ and $\exists B \in \mathcal{I}_{x_q}$ witnessing $P, Q \in \mathbb{P}$, and such that

$$(Q\restriction L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B * \dot{\mathbb{S}}} (P\restriction L_{x_p}, p(x_p)).$$

Completeness of Embeddings Lemma

Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$, let $\mathbb{Q} = \mathbb{Q}_{L_0}$ be the poset for adding m.c.g. with L_0-generators, $\mathbb{S}$ be good σ-Suslin.

Let $B \in \mathcal{I}$, $A \subset B$ be closed. Then $\mathbb{P}_B$ is a poset, $\mathbb{P}_A \subset \mathbb{P}_B$, every $P = (p, F^p) \in \mathbb{P}_B$ has a canonical reduction $P_0 = (p_0, F^{p_0}) \in \mathbb{P}_A$ such that

- $\mathrm{dom}(p_0) = \mathrm{dom}(p) \cap A$, $F^{p_0} = F^p$,
- $s_x^{p_0} = s_x^p$ for all $x \in \mathrm{dom}(p_0) \cap L_1$
- $(p_0 \restriction L_0, F^{p_0})$ is a strong $\mathbb{Q}_A$-reduction of $(p \restriction L_0, F^p)$

and whenever $D \in \mathcal{I}$, $B, C \subseteq D$, C is closed, $C \cap B = A$ and $Q_0 \leq_{\mathbb{P}_C} P_0$, then Q_0 and P are compatible in $\mathbb{P}_D$.

If $A = C$, $D = B$ then $\mathbb{P}_A$ is a complete suborder of $\mathbb{P}_B$.

Lemma

- $\mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ is Knaster.
- Let $x \in L_1$, $A \in \mathcal{I}_x$. Then the two-step iteration $\mathbb{P}_A * \mathbb{S}$ completely embeds into $\mathbb{P}$.
- For any $p \in \mathbb{P}(\mathcal{T}, \mathbb{Q}, \mathbb{S})$ there is countable $A \subseteq L$ such that $p \in \mathbb{P}_{\mathrm{cl}(A)}$. If τ is a $\mathbb{P}$-name for a real then there is a countable $A \subseteq L$ such that τ is a $\mathbb{P}_{\mathrm{cl}(A)}$-name.

Lemma

Let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$ and let λ_0 be a regular uncountable cardinal such that $\lambda_0 \subseteq L_1$ (as an order), λ_0 is cofinal in L, and $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$. Then in $V^{\mathbb{P}}$, non$(\mathcal{M}) = \lambda_0$ and so $\mathfrak{a}_g \geq \lambda_0$.

Proof

Let G be $\mathbb{P}$-generic and let ϕ_α be the slalom added in coordinate $\alpha < \lambda_0$. Since λ_0 is regular, uncountable and is cofinal in L, the family $\langle \phi_\alpha : \alpha < \mu \rangle$ localizes all reals $V[G]$ (indeed any real must appear in some $V[G \cap \mathbb{P}_{L_\alpha}]$ for some $\alpha < \lambda_0$.) Thus $\mathrm{cof}(\mathcal{N}) \leq \lambda_0$. On the other hand, if $F \subseteq \omega^\omega$ is a family of size $< \lambda_0$ in $V[G]$, then there must be some $\alpha < \lambda_0$ such that all reals of F already are in $V[G \cap \mathbb{P}_{L_\alpha}]$, and so ϕ_α localizes all reals in F. Thus $\mathrm{add}(\mathcal{N}) \geq \lambda_0$. Therefore $\mathrm{non}(M) = \lambda_0$ and so $\mathfrak{a}_g \geq \mu$. □

Lemma

Let $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$, L of uncountable cofinality, L_0 cofinal in L. Then $\mathbb{P}$ adds a maximal cofinitary group of size $|L_0|$.

Assume CH. Let $\lambda = \bigcup_n \lambda_n$, where λ_n is a regular cardinal, $\{\lambda_n\}_{n\in\omega}$ increasing and $\lambda_0 \geq \aleph_2$. Consider a template $\mathcal{T} = (L, \mathcal{I})$ such that

- $\lambda_0 \subseteq L_1$, λ_0 is cofinal in L, $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$.
- L has uncountable cofinality, L_0 is cofinal in L.

Then in $V^{\mathbb{P}}$ for $\mathbb{P} = \mathbb{P}(\mathcal{T}, \mathbb{Q}_{L_0}, \mathbb{L})$

- $\lambda_0 = \text{non}(\mathcal{M})$, and so $\lambda_0 \leq \mathfrak{a}_g$
- there is a mcg of size λ and so $\mathfrak{a}_g \leq \lambda$.

An isomorphism of names argument provides that in $V^{\mathbb{P}}$ there are no mcg of size $< \lambda$ and so $V^{\mathbb{P}} \vDash \mathfrak{a}_g = \lambda$.

Theorem (V.F., A. Törnquist)

It is consistent with the usual axioms of set theory that the minimal size of a maximal cofinitary group is of countable cofinality.

Thank you!