Separating club–guessing principles in the presence of fat forcing axioms

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This is joint work with Miguel Angel Mota.

Club Guessing on ω_1

Well–known weakening of Jensen's $\diamondsuit$:

Club Guessing on ω_1 (CG) (Shelah?): There is a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ (i.e., for all δ , $C_\delta \subseteq \delta$ is cofinal in δ and of order type ω) such that for every club $C \subseteq \omega_1$ there is $\delta \in \mathrm{Lim}(\omega_1)$ such that $C_\delta \subseteq_{\mathrm{fin}} C$.

Club Guessing on κ with $\mathrm{cf}(\kappa) \geq \omega_2$ is a ZFC theorem (Shelah).

Some weakenings of CG

Consider the following weakenings of CG:

Kunen's Axiom (KA) (Kunen): There is a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is δ such that

$$[C_\delta(n), C_\delta(n+1)) \cap C \neq \emptyset$$

for a tail of n,
where $(C_\delta(n))_{n<\omega}$ is the increasing enumeration of C_δ.

Clearly: CG $\Longrightarrow$ KA.

$\mho$ (Todorčević, J. Moore): There is a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ and colourings $g_\delta : \delta \longrightarrow \omega$ (for $\delta \in \mathrm{Lim}(\omega_1)$) such that

- For all δ and $n < \omega$, $|g_\delta``(C_\delta(n), C_\delta(n+1)]| = 1$, and
- for every club $C \subseteq \omega_1$ there is some δ such that $g_\delta^{-1}(\{m\}) \cap C$ is unbounded in δ for al $m < \omega$.

Clearly: $\mathrm{KA} \Longrightarrow \mho$.

Weak Club Guessing (WCG) (Shelah): There is a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is δ such that $C_\delta \cap C$ is unbounded in δ.

Very Weak Club Guessing (VWCG) (Shelah): There is a set $\mathcal{X}$ of size $\aleph_1$ consisting of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with a member of $\mathcal{X}$.

Very Weak Club Guessing$_\lambda$ (VWCG$_\lambda$) (A.–Mota): There is a set $\mathcal{X}$ of size $\leq \lambda$ consisting of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with a member of $\mathcal{X}$.

CG $\Longrightarrow$ WCG $\Longrightarrow$ VWCG $=$ VWCG$_{\aleph_1}$

VWCG$_\lambda \Longrightarrow$ VWCG$_\mu$ for $\lambda < \mu$.

$\mathfrak{b} \leq \lambda \Longrightarrow$ VWCG$_\lambda$

The ‘strong’ form of these (weak) guessing principles

We can define these strong forms by requiring that the relevant guessing occurs on a club of δ’s. For example:

Strong Club Guessing (Strong CG): There is a ladder system $(C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there are club–many $\delta \in \mathrm{Lim}(\omega_1)$ such that $C_\delta \subseteq_{\mathrm{fin}} C$.

Similarly we can define strong KA, strong $\mho$, strong weak club guessing, and so on.

Of course Strong P implies P for all these guessing principles P. And the reverse implications don't hold. Also, Strong P_1 implies Strong P_0 if P_1 implies P_0.

Caution: Even if $\diamondsuit$ implies CG, $\diamondsuit^+$ (which is a 'weakly strong' form of $\diamondsuit$) does not imply Strong CG (Ishiu, P. Larson)

These strong guessing principles are consistent (folklore): Add a CG sequence $\vec{C}$ by initial segments. Then do a countable support iteration in which you shoot all relevant clubs to make $\vec{C}$ strongly club guessing.

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Some innocent forcing notions and weak forcing axioms

Given a partial order $\mathcal{P}$ and a cardinal λ, $\mathsf{FA}(\mathbb{P})_\lambda$ means: For every collection $\{D_i \mid i < \lambda\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$.

Given a class Γ of partial orders and a cardinal λ, $\mathsf{FA}(\Gamma)_\lambda$ means $\mathsf{FA}(\mathcal{P})_\lambda$ for every $\mathcal{P} \in \Gamma$.

BPFA implies $\neg$VWCG and $\neg\mho$ (using the natural poset for adding, by initial segments, a club destroying the relevant guessing sequence).

On the other hand, every club of ω_1 in every ccc extension contains a club in V. In particular, all these guessing principles P are preserved by ccc forcing, and so they are consistent with $2^{\aleph_0}$ large.
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Of course MA_{ω_1} implies neither VWCG nor $\mho$, since $\mathrm{BPFA} \implies \mathrm{MA}_{\omega_1}$ and $\mathrm{BPFA} \implies (\neg\mathrm{VWCG} \wedge \neg\mho)$.

What about MA_λ for $\lambda > \omega_1$? Or at least $\mathrm{FA}(\Gamma)_\lambda$ for a reasonable class $\Gamma \subseteq \mathrm{ccc}$?

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Add(ω, θ) always preserves $\neg$CG. On the other hand, Cohen forcing adds a WCG–sequence.

Application: One can always force

$\neg$CG $+$ WCG + Strong KA $+$ $2^{\aleph_0}$ large + FA(Add$(\omega, \lambda))_\mu$ for all λ, $\mu < 2^{\aleph_0}$

(Start with a Strong KA sequence $\vec{C}$. Then force $\neg$CG while preserving that $\vec{C}$ is a strong KA sequence with a suitable countable support proper forcing iteration. Then add many Cohen reals.)

In fact one can get

$\neg$CG $+$ $\mathfrak{b} = \omega_1$ + Strong KA $+$ $2^{\aleph_0}$ large + FA(Add$(\omega, \lambda))_\mu$ for all λ, $\mu < 2^{\aleph_0}$.

$\mathrm{Add}(\omega, \theta)$ always preserves $\neg\mathsf{CG}$. On the other hand, Cohen forcing adds a WCG–sequence.

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$\neg\mathsf{CG} + \mathsf{WCG}$ + Strong $\mathsf{KA} + 2^{\aleph_0}$ large + $\mathsf{FA}(\mathrm{Add}(\omega, \lambda))_\mu$ for all λ, $\mu < 2^{\aleph_0}$

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For every λ, ${}^{\omega}\omega$–bounding forcing preserves $\neg$WCG and $\neg$VWCG$_\lambda$.

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Strong KA $+ \neg$VWCG $+ 2^{\aleph_0}$ large + FA$(\lambda\text{–randoms})_\mu$ for all λ, $\mu < 2^{\aleph_0}$

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Two *natural* questions at this point

What about showing MA_λ, for large λ, consistent with $\neg P$ for some / all of our guessing principles P? (Note that any long enough finite support c.c.c. iteration will force WCG since it adds a Cohen real over V at stage ω, and therefore a WCG–sequence which will remain WCG in the end.)

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Extending Martin's Axiom

Definition (A.–Mota): A poset $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if there is a decomposition $\mathcal{P} = \bigcup_{\nu<\omega_1} P_\nu$ such that for all ν, $p \in P_\nu$ and all countable elementary substructures $N_0, \ldots N_n \preccurlyeq H(\theta)$ containing $\mathcal{P}$, $\theta > |\mathcal{P}|$, if $\nu \in N_i \cap \omega_1$ for all $i \leq n$, then there is $q \leq_{\mathcal{P}} p$, q $(N_i, \mathcal{P})$–generic for all i.

$\aleph_1$–c.c. $\subseteq$ $\aleph_{1.5}$–c.c. $\subseteq$ $\aleph_2$–c.c.

$\aleph_1$–c.c. $\subseteq$ finitely proper $\subseteq$ proper.

If $|\mathcal{P}| = \aleph_1$, then $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if and only if $\mathcal{P}$ is finitely proper.

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If $|\mathcal{P}| = \aleph_1$, then $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if and only if $\mathcal{P}$ is finitely proper.

Definition (A.–Mota): $\mathrm{MA}^{1.5}_{\lambda}$ is $\mathrm{FA}(\aleph_{1.5}\text{–c.c.})_{\lambda}$.

Theorem 1 (A.–Mota): Suppose CH holds. Let $\kappa \geq \omega_3$ be a regular cardinal such that $\mu^{\aleph_1} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa \mid cf(\alpha) \geq \omega_2\})$ holds. Then there exists a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–c.c. such that the following statements hold in the generic extension by $\mathcal{P}$:

(1) $2^{\aleph_0} = \kappa$

(2) $\mathrm{MA}^{1.5}_{\lambda}$ for every $\lambda < 2^{\aleph_0}$.

The proof of Theorem 1 is by a finite support iteration with (partial) homogeneous systems of countable structures as side conditions.

A prominent $\aleph_{1.5}$–c.c. forcing

$\mathbb{B}$: Baumgartner's forcing for adding a club of ω_1 with finite conditions:
Conditions are finite functions $p \subseteq \omega_1 \times \omega_1$ such that p can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \omega_1$.

$\mathbb{B}$ is $\aleph_{1.5}$–c.c. (in fact, finitely proper and of size $\aleph_1$).

$\mathbb{B}$ adds a generic for $\mathrm{Add}(\omega, \omega_1)$.

Zapletal: (PFA) Every nowhere ccc poset (i.e., not ccc below any condition) of size $\aleph_1$ adds a generic for $\mathbb{B}$.

Definition: A set $\mathcal{C}$ of subsets of ω_1 of order type ω is a KA set if for every club $D \subseteq \omega_1$ there is some $C \in \mathcal{C}$ such that $D \cap [C(n), C(n+1)) \neq \emptyset$ for a tail of $n < \omega$.

$\mathbb{B}$ destroys every KA–sequence from the ground model. In particular, $\mathsf{FA}(\mathbb{B})_\lambda$ implies there are no KA sets of size $\leq \lambda$, and hence Theorem 1 shows the consistency of

MA + $2^{\aleph_0}$ large + There are no KA sets of size $< 2^{\aleph_0}$.

Another application of $\mathrm{MA}^{1.5}_{\lambda}$

Also: $\mathrm{MA}^{1.5}_{\lambda}$ implies $\neg\mathrm{VWCG}_{\lambda}$.

Given a potential VWCG_{λ} set $\mathcal{X}$, the forcing for this consists of conditions of Baumgartner's forcing together with finite sets of promises of avoiding certain co-finite subsets of finitely members from $\mathcal{X}$.

Hence,Theorem 1 shows in fact the consistency of

$\mathrm{MA} + 2^{\aleph_0}$ large + $\neg\mathrm{VWCG}_{\lambda}$ for all $\lambda < 2^{\aleph_0}$.

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$\mathrm{MA} + 2^{\aleph_0}$ large + $\neg\mathrm{VWCG}_\lambda$ for all $\lambda < 2^{\aleph_0}$.

Separating guessing principles in the presence of fragments of MA[1.5]

Theorem 2 (A.–Mota): Suppose CH holds and suppose there is a strong $\mho$–sequence $\vec{C}$. Let κ be a regular cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–c.c. such that the following statements hold in $V^{\mathcal{P}}$.

(1) $\vec{C}$ is a strong $\mho$–sequence.

(2) $\neg\mathrm{VWCG}_\lambda$ for all $\lambda < 2^{\aleph_0}$.

(3) MA

(4) $\mathrm{FA}(\mathbb{B})_\lambda$ for all $\lambda < 2^{\aleph_0}$. In particular, there are no KA sets of size $< 2^{\aleph_0}$.

(5) $2^{\aleph_0} = \kappa$

Theorem 3 (A.–Mota): Suppose CH holds and suppose there is a strong WCG–sequence $\vec{C}$. Let κ be a regular cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition such that the following statements hold in $V^{\mathcal{P}}$.

(1) $\vec{C}$ is a strong WCG–sequence.

(2) $\neg\mho$

(3) MA

(4) $\mathrm{FA}(\mathbb{B})_\lambda$ for all $\lambda < 2^{\aleph_0}$. In particular, there are no KA sets of size $< 2^{\aleph_0}$.

(5) $2^{\aleph_0} = \kappa$

Theorems 2 and 3 have similar proofs, but the proof of Theorem 2 doesn't need to use predicates (see below).

Rough proof sketch of Theorem 3:
Suppose $\vec{C} = (C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ is a strong WCG–sequence. We build $\mathcal{P} = \mathcal{P}_\kappa$, where $(\mathcal{P}_\alpha \mid \alpha \leq \kappa)$ is a certain finite support iteration with "homogeneous systems of countable structures **with predicates**" as side conditions.

Conditions of $\mathcal{P}_\alpha$: pairs of the form $q = (F, \Delta)$, where

(1) F is a α–sequence with finite support giving finite information on the relevant tasks specified by some book-keeping (killing instances of $\mho$, shooting clubs to preserve that $\vec{C}$ is strongly WCG, and forcing with $\mathbb{B}$ and with c.c.c. posets).

(2) $\Delta = \{(N_i, \vec{\mathcal{W}}^i, \gamma_i) \mid i < n\}$, where
   - $\{N_i \mid i < n\}$ is a finite 'homogeneous' system of elementary substructures of $H(\kappa)$,
   - $\gamma_i \leq \min\{\alpha, \sup(N_i \cap \kappa)\}$, and
   - $\vec{\mathcal{W}}^i = (W^i_m)_{m<\omega}$ and for all m, $W^i_m \subseteq N_i$ and W^i_m consists of pairs $(M, \vec{\mathcal{V}})$, etc., such that $M \cap \omega_1 \in C_{N_i \cap \omega_1}$.

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The side condition specification at stage $\alpha+1$:

If $(N,(W_m)_{m<\omega},\alpha+1)\in\Delta$ and $\alpha+1\in N$, then

$$q|_\alpha=(F\restriction\alpha,\{(N_i,\vec{\mathcal{W}}^i,\min\{\gamma_i,\alpha\})\mid(N_i,\vec{\mathcal{W}}^i,\gamma_i)\in\Delta\})$$

forces in $\mathcal{P}_\alpha$:

(a) For all $m<\omega$, the set

$$\mathcal{Y}=\{(M,\vec{\mathcal{V}})\in W_m\mid(M,\vec{\mathcal{V}},\alpha)\in\Delta_r\text{ for some }r\in\dot{G}_\alpha\}$$

is "N–large", in the sense that for every $x\in N$ there is some $(M,\vec{\mathcal{V}})\in\mathcal{Y}$ such that $x\in M$.

(b) If α is in the support of F, then $q|_\alpha$ forces that $F(\alpha)$ is $(N[\dot{G}_\alpha],\dot{\mathcal{Q}}_\alpha)$–proper, for the relevant forcing $\dot{\mathcal{Q}}_\alpha$ picked at stage α.

One proves the relevant facts about $(\mathcal{P}_\alpha \mid \alpha \leq \kappa)$.

All proofs are quite standard except for the proof of properness.

The proof of properness is by induction on α: One proves that if $N \in \mathcal{M}_{\alpha+1}$, where $\mathcal{M}_{\alpha+1}$ is a club of countable $M \subseteq H(\kappa)$ such that $(M, \in, \mathcal{P}_\alpha \cap M) \prec (H(\kappa), \in, \mathcal{P}_\alpha)$, and $q = (F, \Delta) \in \mathcal{P}_\alpha \cap N$, then there is $\vec{\mathcal{W}}$ such that

$$(F', \Delta \cup \{(N, \vec{\mathcal{W}}, \alpha)\})$$

is $(N, \mathcal{P}_\alpha)$–generic, where F' is easily constructed from F. The homogeneity of the side conditions is used only in the case $\mathrm{cf}(\alpha) > \omega$ of the induction. The fact that $\vec{C}$ is strongly WCG is used. We don't know how to prove the theorem if we assume $\vec{C}$ is just WCG.

End of proof sketch. $\square$

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End of proof sketch. $\square$

What about higher cardinalities?

Observation: (GCH) Given a regular $\kappa \geq \omega$, there is a $<\kappa$–directed closed forcing which is proper with respect to internally approachable elementary substructures of size κ and which forces that for every club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ there is a club $D \subseteq \kappa^+$ such that for all $\delta \in D \cap \mathrm{cf}(\kappa)$ there are stationarily many $\alpha < \mathrm{ot}(C_\delta)$ such that $(C_\delta(\alpha),\ C_\delta(\alpha + 1)] \cap D = \emptyset$.

(Proof: Do a κ–support κ^+–iteration adding clubs of κ^+ by approximations of size $<\kappa$. No iteration theory is needed to prove the relevant properness.)

On the other hand:

Theorem (Shelah): For every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all κ and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in \kappa^+ \cap \mathrm{cf}(\kappa)$ such that $C_\delta(\alpha + 1) \in D$ for stationarily many $\alpha < \kappa$.

Given a club–sequence $\vec{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \mathrm{cf}(\kappa)\rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all κ there is a forcing for destroying the above guessing property of $\vec{C}$ and which is $<\kappa$–directed closed and proper with respect to internally approachable elementary structures of size κ. The above theorem of course shows that there can be no iteration theory for this version of high properness.

Thank you!