Separating club–guessing principles in the presence of fat forcing axioms

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This is joint work with Miguel Angel Mota.
Well–known weakening of Jensen’s ♠:

Club Guessing on \( \omega_1 \) (CG) (Shelah?): There is a ladder system \( (C_\delta \mid \delta \in \text{Lim}(\omega_1)) \) (i.e., for all \( \delta \), \( C_\delta \subseteq \delta \) is cofinal in \( \delta \) and of order type \( \omega \)) such that for every club \( C \subseteq \omega_1 \) there is \( \delta \in \text{Lim}(\omega_1) \) such that \( C_\delta \subseteq \text{fin} \ C \).

Club Guessing on \( \kappa \) with \( \text{cf}(\kappa) \geq \omega_2 \) is a ZFC theorem (Shelah).
Some weakenings of CG

Consider the following weakenings of CG:

Kunen’s Axiom (KA) (Kunen): There is a ladder system
\( (C_\delta \mid \delta \in \text{Lim}(\omega_1)) \) such that for every club \( C \subseteq \omega_1 \) there is \( \delta \) such that

\[
[C_\delta(n), \; C_\delta(n + 1)) \cap C \neq \emptyset
\]

for a tail of \( n \),
where \( (C_\delta(n))_{n<\omega} \) is the increasing enumeration of \( C_\delta \).

Clearly: \( \text{CG} \implies \text{KA} \).
U (Todorčević, J. Moore): There is a ladder system \((C_\delta \mid \delta \in \text{Lim}(\omega_1))\) and colourings \(g_\delta : \delta \longrightarrow \omega\) (for \(\delta \in \text{Lim}(\omega_1)\)) such that

- For all \(\delta\) and \(n < \omega\), \(|g_\delta^{-1}(\{C_\delta(n), C_\delta(n+1)\})| = 1\), and
- for every club \(C \subseteq \omega_1\) there is some \(\delta\) such that \(g_\delta^{-1}(\{m\}) \cap C\) is unbounded in \(\delta\) for all \(m < \omega\).

Clearly: \(\text{KA} \Rightarrow U\).
Weak Club Guessing (WCG) (Shelah): There is a ladder system \( (C_\delta \mid \delta \in \text{Lim}(\omega_1)) \) such that for every club \( C \subseteq \omega_1 \) there is \( \delta \) such that \( C_\delta \cap C \) is unbounded in \( \delta \).

Very Weak Club Guessing (VWCG) (Shelah): There is a set \( \mathcal{X} \) of size \( \aleph_1 \) consisting of subsets of \( \omega_1 \) of order type \( \omega \) such that every club of \( \omega_1 \) has infinite intersection with a member of \( \mathcal{X} \).

Very Weak Club Guessing_\lambda (VWCG_\lambda) (A.–Mota): There is a set \( \mathcal{X} \) of size \( \leq \lambda \) consisting of subsets of \( \omega_1 \) of order type \( \omega \) such that every club of \( \omega_1 \) has infinite intersection with a member of \( \mathcal{X} \).
CG $\rightarrow$ WCG $\rightarrow$ VWCG = VWCG\(_{\lambda_1}\)

VWCG\(\lambda\) $\Rightarrow$ VWCG\(\mu\) for \(\lambda < \mu\).

\(b \leq \lambda \Rightarrow VWCG\lambda\)
The ‘strong’ form of these (weak) guessing principles

We can define these strong forms by requiring that the relevant guessing occurs on a club of \( \delta \)'s. For example:

Strong Club Guessing (Strong CG): There is a ladder system \( (C_\delta \mid \delta \in \text{Lim}(\omega_1)) \) such that for every club \( C \subseteq \omega_1 \) there are club–many \( \delta \in \text{Lim}(\omega_1) \) such that \( C_\delta \subseteq_{\text{fin}} C \).

Similarly we can define strong KA, strong \( \mathcal{U} \), strong weak club guessing, and so on.
Of course Strong $P$ implies $P$ for all these guessing principles $P$. And the reverse implications don’t hold. Also, Strong $P_1$ implies Strong $P_0$ if $P_1$ implies $P_0$.

**Caution:** Even if $\diamond$ implies CG, $\diamond^+$ (which is a ‘weakly strong’ form of $\diamond$) does not imply Strong CG (Ishiu, P. Larson)

These strong guessing principles are consistent (folklore): Add a CG sequence $\tilde{C}$ by initial segments. Then do a countable support iteration in which you shoot all relevant clubs to make $\tilde{C}$ strongly club guessing.
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These strong guessing principles are consistent (folklore): Add a CG sequence $\bar{C}$ by initial segments. Then do a countable support iteration in which you shoot all relevant clubs to make $\bar{C}$ strongly club guessing.
Some innocent forcing notions and weak forcing axioms

Given a partial order $\mathcal{P}$ and a cardinal $\lambda$, $\text{FA}(\mathcal{P})_{\lambda}$ means: For every collection $\{\mathcal{D}_i \mid i < \lambda\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap \mathcal{D}_i \neq \emptyset$ for all $i < \lambda$.

Given a class $\Gamma$ of partial orders and a cardinal $\lambda$, $\text{FA}(\Gamma)_{\lambda}$ means $\text{FA}(\mathcal{P})_{\lambda}$ for every $\mathcal{P} \in \Gamma$. 
BPFA implies $\neg\text{VWCG}$ and $\neg\mathcal{U}$ (using the natural poset for adding, by initial segments, a club destroying the relevant guessing sequence).

On the other hand, every club of $\omega_1$ in every ccc extension contains a club in $V$. In particular, all these guessing principles $P$ are preserved by ccc forcing, and so they are consistent with $2^{\aleph_0}$ large.
In particular, no forcing axiom $\text{MA}_\lambda$ implies $\neg\text{Strong CG}$. 
BPFA implies \( \neg \text{VWCG} \) and \( \neg \mathcal{U} \) (using the natural poset for adding, by initial segments, a club destroying the relevant guessing sequence).

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In particular, no forcing axiom MA\( _\lambda \) implies \( \neg \text{Strong CG} \).
BPFA implies $\neg$VWCG and $\neg$reachable (using the natural poset for adding, by initial segments, a club destroying the relevant guessing sequence).

On the other hand, every club of $\omega_1$ in every ccc extension contains a club in $V$. In particular, all these guessing principles $P$ are preserved by ccc forcing, and so they are consistent with $2^{\aleph_0}$ large. In particular, no forcing axiom $\text{MA}_\lambda$ implies $\neg$ Strong CG.
Of course $\text{MA}_{\omega_1}$ implies neither VWCG nor $\mathcal{U}$, since $\text{BPFA} \iff \text{MA}_{\omega_1}$ and $\text{BPFA} \iff (\neg \text{VWCG} \land \neg \mathcal{U})$.

What about $\text{MA}_{\lambda}$ for $\lambda > \omega_1$? Or at least $\text{FA}(\Gamma)_\lambda$ for a reasonable class $\Gamma \subseteq \text{ccc}$?
Of course $\text{MA}_{\omega_1}$ implies neither $\text{VWCG}$ nor $\mathcal{U}$, since $\text{BPFA} \iff \text{MA}_{\omega_1}$ and $\text{BPFA} \iff (\neg \text{VWCG} \land \neg \mathcal{U})$.

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Add(ω, θ) always preserves ¬CG. On the other hand, Cohen forcing adds a WCG–sequence.

Application: One can always force

$$\neg CG + WCG + \text{Strong KA} + 2^{\aleph_0} \text{ large} + \text{FA}(\text{Add}(\omega, \lambda))_\mu \text{ for all } \lambda, \mu < 2^{\aleph_0}$$

(Start with a Strong KA sequence \(\tilde{C}\). Then force \(\neg CG\) while preserving that \(\tilde{C}\) is a strong KA sequence with a suitable countable support proper forcing iteration. Then add many Cohen reals.)

In fact one can get

$$\neg CG + b = \omega_1 + \text{Strong KA} + 2^{\aleph_0} \text{ large} + \text{FA}(\text{Add}(\omega, \lambda))_\mu \text{ for all } \lambda, \mu < 2^{\aleph_0}.$$
Add(\(\omega, \theta\)) always preserves \(\neg\text{CG}\). On the other hand, Cohen forcing adds a WCG–sequence.

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In fact one can get

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\]
For every $\lambda$, $\omega_\omega$–bounding forcing preserves $\neg WCG$ and $\neg VWCG_\lambda$.

Application: One can always force

Strong $KA + \neg VWCG + 2^{\aleph_0}$ large + $\text{FA}(\lambda\text{–randoms})_\mu$ for all $\lambda$, $\mu < 2^{\aleph_0}$

(Start with Strong $KA + \neg VWCG$, which can be forced in a similar way as before, and add lots of random reals.)
For every $\lambda$, $\omega_\omega$–bounding forcing preserves $\neg WCG$ and $\neg VWCG_\lambda$.

Application: One can always force

Strong $KA + \neg VWCG + 2^{\aleph_0}$ large $+ FA(\lambda$–randoms)$_\mu$ for all $\lambda$, $\mu < 2^{\aleph_0}$

(Start with Strong $KA + \neg VWCG$, which can be forced in a similar way as before, and add lots of random reals.)
Two *natural* questions at this point

What about showing $\text{MA}_\lambda$, for large $\lambda$, consistent with $\neg P$ for some / all of our guessing principles $P$? (Note that any long enough finite support c.c.c. iteration will force WCG since it adds a Cohen real over $V$ at stage $\omega$, and therefore a WCG–sequence which will remain WCG in the end.)

What about forcing $\neg\text{VWCG}_\lambda$ for any $\lambda > \omega_1$? (Maybe $\text{VWCG}_{\aleph_2}$ is a ZFC theorem?)
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Definition (A.–Mota): A poset $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if there is a decomposition $\mathcal{P} = \bigcup_{\nu < \omega_1} P_\nu$ such that for all $\nu$, $p \in P_\nu$ and all countable elementary substructures $N_0, \ldots N_n \prec H(\theta)$ containing $\mathcal{P}$, $\theta > |\mathcal{P}|$, if $\nu \in N_i \cap \omega_1$ for all $i \leq n$, then there is $q \leq_{\mathcal{P}} p$, $q (N_i, \mathcal{P})$–generic for all $i$.

$\aleph_1$–c.c. $\subseteq$ $\aleph_{1.5}$–c.c. $\subseteq$ $\aleph_2$–c.c.

$\aleph_1$–c.c. $\subseteq$ finitely proper $\subseteq$ proper.

If $|\mathcal{P}| = \aleph_1$, then $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if and only if $\mathcal{P}$ is finitely proper.
Extending Martin’s Axiom

Definition (A.–Mota): A poset $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if there is a decomposition $\mathcal{P} = \bigcup_{\nu < \omega_1} P_\nu$ such that for all $\nu$, $p \in P_\nu$ and all countable elementary substructures $N_0, \ldots N_n \subseteq H(\theta)$ containing $\mathcal{P}$, $\theta > |\mathcal{P}|$, if $\nu \in N_i \cap \omega_1$ for all $i \leq n$, then there is $q \leq_\mathcal{P} p$, $q (N_i, \mathcal{P})$–generic for all $i$.

$\aleph_1$–c.c. $\subseteq \aleph_{1.5}$–c.c. $\subseteq \aleph_2$–c.c.

$\aleph_1$–c.c. $\subseteq$ finitely proper $\subseteq$ proper.

If $|\mathcal{P}| = \aleph_1$, then $\mathcal{P}$ is $\aleph_{1.5}$–c.c. if and only if $\mathcal{P}$ is finitely proper.
Definition (A.–Mota): $\text{MA}_{\lambda}^{1.5}$ is $\text{FA}(\aleph_{1.5} \text{-c.c.})_{\lambda}$.

Theorem 1 (A.–Mota): Suppose CH holds. Let $\kappa \geq \omega_3$ be a regular cardinal such that $\mu^\aleph_1 < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa \mid \text{cf}(\alpha) \geq \omega_2\})$ holds. Then there exists a proper forcing notion $\mathcal{P}$ of size $\kappa$ with the $\aleph_2$–c.c. such that the following statements hold in the generic extension by $\mathcal{P}$:

1. $2^{\aleph_0} = \kappa$
2. $\text{MA}_{\lambda}^{1.5}$ for every $\lambda < 2^{\aleph_0}$.

The proof of Theorem 1 is by a finite support iteration with (partial) homogeneous systems of countable structures as side conditions.
A prominent $\aleph_{1.5}$–c.c. forcing

$\mathcal{B}$: Baumgartner’s forcing for adding a club of $\omega_1$ with finite conditions:
Conditions are finite functions $p \subseteq \omega_1 \times \omega_1$ such that $p$ can be extended to a strictly increasing and continuous function $F : \omega_1 \rightarrow \omega_1$.

$\mathcal{B}$ is $\aleph_{1.5}$–c.c. (in fact, finitely proper and of size $\aleph_1$).

$\mathcal{B}$ adds a generic for $\text{Add}(\omega, \omega_1)$.

Zapletal: (PFA) Every nowhere ccc poset (i.e., not ccc below any condition) of size $\aleph_1$ adds a generic for $\mathcal{B}$. 
Definition: A set $C$ of subsets of $\omega_1$ of order type $\omega$ is a KA set if for every club $D \subseteq \omega_1$ there is some $C \in C$ such that $D \cap [C(n), C(n + 1)) \neq \emptyset$ for a tail of $n < \omega$.

$B$ destroys every KA–sequence from the ground model. In particular, $\text{FA}(B)_\lambda$ implies there are no KA sets of size $\leq \lambda$, and hence Theorem 1 shows the consistency of

$\text{MA} + 2^{\aleph_0}$ large + There are no KA sets of size $< 2^{\aleph_0}$. 

Another application of $\text{MA}^{1.5}_\lambda$

Also: $\text{MA}^{1.5}_\lambda$ implies $\neg\text{VWCG}_\lambda$.

Given a potential $\text{VWCG}_\lambda$ set $\mathcal{X}$, the forcing for this consists of conditions of Baumgartner’s forcing together with finite sets of promises of avoiding certain co-finite subsets of finitely members from $\mathcal{X}$.

Hence, Theorem 1 shows in fact the consistency of $\text{MA} + 2^{\aleph_0} \text{ large} + \neg\text{VWCG}_\lambda$ for all $\lambda < 2^{\aleph_0}$. 
Another application of $\text{MA}^{1.5}_\lambda$

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Hence, Theorem 1 shows in fact the consistency of

$\text{MA} + 2^{\aleph_0}$ large + $\neg\text{VWCG}_\lambda$ for all $\lambda < 2^{\aleph_0}$.
Separating guessing principles in the presence of fragments of $\text{MA}^{1.5}$

**Theorem 2 (A.–Mota):** Suppose $\text{CH}$ holds and suppose there is a strong $\mathcal{U}$–sequence $\vec{C}$. Let $\kappa$ be a regular cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–c.c. such that the following statements hold in $V^{\mathcal{P}}$.

1. $\vec{C}$ is a strong $\mathcal{U}$–sequence.
2. $\neg \text{VWCG}_\lambda$ for all $\lambda < 2^{\aleph_0}$.
3. $\text{MA}$
4. $\text{FA}(\mathcal{B})_\lambda$ for all $\lambda < 2^{\aleph_0}$. In particular, there are no KA sets of size $< 2^{\aleph_0}$.
5. $2^{\aleph_0} = \kappa$
Theorem 3 (A.–Mota): Suppose CH holds and suppose there is a strong WCG–sequence $\tilde{C}$. Let $\kappa$ be a regular cardinal such that $\kappa^{\aleph_1} = \kappa$ and $2^{<\kappa} = \kappa$. Then there exists a proper poset $\mathcal{P}$ with the $\aleph_2$–chain condition such that the following statements hold in $V^\mathcal{P}$.

1. $\tilde{C}$ is a strong WCG–sequence.
2. $\neg \mathfrak{D}$
3. MA
4. $\text{FA}(\mathbb{B})_\lambda$ for all $\lambda < 2^{\aleph_0}$. In particular, there are no KA sets of size $< 2^{\aleph_0}$.
5. $2^{\aleph_0} = \kappa$
Theorems 2 and 3 have similar proofs, but the proof of Theorem 2 doesn’t need to use predicates (see below).

Rough proof sketch of Theorem 3:
Suppose $\tilde{C} = (C_\delta \mid \delta \in \text{Lim}(\omega_1))$ is a strong WCG–sequence. We build $\mathcal{P} = \mathcal{P}_\kappa$, where $(\mathcal{P}_\alpha \mid \alpha \leq \kappa)$ is a certain finite support iteration with “homogeneous systems of countable structures with predicates” as side conditions.
Conditions of $\mathcal{P}_\alpha$: pairs of the form $q = (F, \Delta)$, where

1. $F$ is a $\alpha$–sequence with finite support giving finite information on the relevant tasks specified by some book-keeping (killing instances of $\mathcal{U}$, shooting clubs to preserve that $\tilde{C}$ is strongly WCG, and forcing with $\mathbb{B}$ and with c.c.c. posets).

2. $\Delta = \{(N_i, \tilde{W}^i, \gamma_i) \mid i < n\}$, where
   - $\{N_i \mid i < n\}$ is a finite ‘homogeneous’ system of elementary substructures of $H(\kappa)$,
   - $\gamma_i \leq \min\{\alpha, \sup(N_i \cap \kappa)\}$, and
   - $\tilde{W}^i = (W^i_m)_{m<\omega}$ and for all $m$, $W^i_m \subseteq N_i$ and $W^i_m$ consists of pairs $(M, \tilde{V})$, etc., such that $M \cap \omega_1 \in C_{N_i \cap \omega_1}$. 
Conditions of $\mathcal{P}_\alpha$: pairs of the form $q = (F, \Delta)$, where

1. $F$ is a $\alpha$–sequence with finite support giving finite information on the relevant tasks specified by some book-keeping (killing instances of $U$, shooting clubs to preserve that $\mathcal{C}$ is strongly WCG, and forcing with $B$ and with c.c.c. posets).

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   - $\{N_i \mid i < n\}$ is a finite ‘homogeneous’ system of elementary substructures of $H(\kappa)$,
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The side condition specification at stage $\alpha + 1$:

If $(N, (W_m)_{m<\omega}, \alpha + 1) \in \Delta$ and $\alpha + 1 \in N$, then

$$q|\alpha = (F \upharpoonright \alpha, \{(N_i, \vec{W}_i, \min\{\gamma_i, \alpha\}) | (N_i, \vec{W}_i, \gamma_i) \in \Delta\})$$

forces in $\mathcal{P}_\alpha$:

(a) For all $m < \omega$, the set

$$\mathcal{Y} = \{(M, \vec{V}) \in W_m | (M, \vec{V}, \alpha) \in \Delta_r \text{ for some } r \in \dot{G}_\alpha\}$$

is “$N$–large”, in the sense that for every $x \in N$ there is some $(M, \vec{V}) \in \mathcal{Y}$ such that $x \in M$.

(b) If $\alpha$ is in the support of $F$, then $q|\alpha$ forces that $F(\alpha)$ is $(N[\dot{G}_\alpha], \dot{Q}_\alpha)$–proper, for the relevant forcing $\dot{Q}_\alpha$ picked at stage $\alpha$. 
One proves the relevant facts about \((\mathcal{P}_\alpha \mid \alpha \leq \kappa)\).

All proofs are quite standard except for the proof of properness.

The proof of properness is by induction on \(\alpha\): One proves that if \(N \in \mathcal{M}_{\alpha+1}\), where \(\mathcal{M}_{\alpha+1}\) is a club of countable \(M \subseteq H(\kappa)\) such that \((M, \in, \mathcal{P}_\alpha \cap M) \prec (H(\kappa), \in, \mathcal{P}_\alpha)\), and \(q = (F, \Delta) \in \mathcal{P}_\alpha \cap N\), then there is \(\tilde{W}\) such that

\[
(F', \Delta \cup \{(N, \tilde{W}, \alpha)\})
\]

is \((N, \mathcal{P}_\alpha)\)-generic, where \(F'\) is easily constructed from \(F\). The homogeneity of the side conditions is used only in the case \(\text{cf}(\alpha) > \omega\) of the induction. The fact that \(\tilde{C}\) is strongly WCG is used. We don’t know how to prove the theorem if we assume \(\tilde{C}\) is just WCG.

End of proof sketch. \(\Box\)
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End of proof sketch. \(\square\)
What about higher cardinalities?

**Observation**: (GCH) Given a regular $\kappa \geq \omega$, there is a $<\kappa$–directed closed forcing which is proper with respect to internally approachable elementary substructures of size $\kappa$ and which forces that for every club–sequence $\langle C_{\delta} \mid \delta \in \kappa^+ \cap \text{cf}(\kappa) \rangle$ there is a club $D \subseteq \kappa^+$ such that for all $\delta \in D \cap \text{cf}(\kappa)$ there are stationarily many $\alpha < \text{ot}(C_{\delta})$ such that $(C_{\delta}(\alpha), C_{\delta}(\alpha + 1)) \cap D = \emptyset$.

(Proof: Do a $\kappa$–support $\kappa^+$–iteration adding clubs of $\kappa^+$ by approximations of size $<\kappa$. No iteration theory is needed to prove the relevant properness.)
On the other hand:

**Theorem** (Shelah): For every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\langle C_\delta \mid \delta \in \kappa^+ \cap \text{cf}(\kappa) \rangle$ with $\text{ot}(C_\delta) = \kappa$ for all $\kappa$ and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in \kappa^+ \cap \text{cf}(\kappa)$ such that $C_\delta(\alpha + 1) \in D$ for stationarily many $\alpha < \kappa$.

Given a club–sequence $\tilde{C} = \langle C_\delta \mid \delta \in \kappa^+ \cap \text{cf}(\kappa) \rangle$ with $\text{ot}(C_\delta) = \kappa$ for all $\kappa$ there is a forcing for destroying the above guessing property of $\tilde{C}$ and which is $<\kappa$–directed closed and proper with respect to internally approachable elementary structures of size $\kappa$. The above theorem of course shows that there can be no iteration theory for this version of high properness.
Thank you!